Dimension-Transcendent Attractors in Generalized Kaprekar Routines

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Abstract

The classical Kaprekar routine at four digits iterates $K(n) = \alpha(n) - \beta(n)$, where $\alpha(n)$ and $\beta(n)$ are the integers formed by arranging n's digits in descending and ascending order. For every non-repdigit four-digit input, the orbit converges in at most seven steps to the fixed point 6174. Generalizing the construction to arbitrary digit length d, the analogous rule converges to 495 at $d = 3$ but fails at $d = 5$: orbits enter cycles rather than fixed points.

This failure is commonly described as a peculiarity of five-digit arithmetic. We argue that it is instead a peculiarity of the specific subtraction rule. The classical "descending minus ascending" construction is one specific point in a space of $d! \cdot (d! - 1)$ ordered permutation-pair rules at each digit length, and its behavior across d is not uniform: the algebraic rank of the classical rule is d at $d = 4$ but strictly less than d at $d = 3$ and $d = 5$ because of forced middle-digit cancellation. The failure at $d = 5$ reflects the classical rule being rank-4 there, not a failure of Kaprekar convergence.

When we classify all permutation-pair rules at each digit length, universal full-variable fixed points exist at $d = 4$, $d = 5$, and $d = 6$. We complete this classification exhaustively at $d \leq 6$, identifying 33 universal fixed points at $d = 5$ and 506 at $d = 6$. We then ask which fixed points extend from one digit length to the next: if F is a universal full-variable fixed point at digit length d, does some rule at $d + 1$ also have F as a universal fixed point?

Among the 33 universal fixed points at $d = 5$, exactly one — $F = 60714$ — is a universal full-variable fixed point at $d = 6$. The remaining 32 are dimension-locked at $d = 5$: no full-variable rule at $d = 6$ attracts them universally. The selection of 60714 from this population is the central discovery; the lifting machinery is the test it uniquely passes, not a construction tailored to fit it. Given this selection, we construct an explicit family of permutation-pair rules at each $d \geq 5$, connected by a coefficient-preserving lifting recipe, and prove:

Theorem. *60714 is a fixed point at every digit length $d \geq 5$ under the coefficient-preserving lifting family constructed in §5. The corresponding dynamics are universal on $A_d \setminus E_d$ at every $d \geq 5$, where A_d is the admissible input set and E_d is an exactly-characterized escape class. The escape class is empty at $d = 5, 6$, and at $d \geq 7$ has the same structural form — block-aligned multisets that collapse to 0 under the rule — as the classical Kaprekar escape class at $d = 3, 4$ (the ten repdigits at $d = 3$ and the ten multiples of 1111 at $d = 4$, which under the classical rule each map to 0 rather than to the nontrivial fixed point).*

This is, to our knowledge, the first explicitly constructed attractor in the generalized Kaprekar family for which a fixed-point equation persists across infinitely many dimensions, with universal convergence on $A_d \setminus E_d$ at every $d \geq 5$ and a structurally characterized escape class continuous with the classical case (§7.6). The classical Kaprekar constant 6174 exhibits a related but distinct pattern, proven here: universal at $d = 4$ (native), algebraically obstructed at $d = 5$,

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and universal on $A_d \setminus E_d^{6174}$ at every $d \geq 6$ under coefficient-preserving lifting, where E_d^{6174} is a precisely characterized 45-input class at $d \geq 8$. 60714, 6174, and 60417 share the digit multiset $\{7, 6, 4, 1\}$, and within this thread the empirical invariant sum_locked_spans orders cross-dimensional behavior: 60714 (sls $= 5$) is universal on $A_d \setminus E_d$ at every $d \geq 5$ with $E_d = \emptyset$ at $d = 5, 6$, while 6174 (sls $= 8$) is universal on $A_d \setminus E_d^{6174}$ at every $d \geq 6$. The invariant orders within-thread behaviors but does not generalize to a cross-multiset principle.

1 Introduction

1.1 Kaprekar's routine and the middle-digit cancellation

In 1949, D. R. Kaprekar observed a striking property of four-digit integers in base ten [Kaprekar 1955]. For any four-digit n with at least two distinct digits, let $\alpha(n)$ be the integer formed by arranging n's digits in descending order, and $\beta(n)$ the integer formed by arranging them in ascending order. Iterating

$$K(n) = \alpha(n) - \beta(n)$$

produces a sequence that converges to 6174 within at most seven steps, regardless of starting value.

At three digits, the analogous construction converges to 495 [Trigg 1974]. At five digits, it fails: every non-repdigit orbit enters one of three cycles rather than reaching a fixed point [Prichett 1981]. This failure is typically described as a peculiarity of five-digit arithmetic. We argue that it reveals something deeper about the classical rule itself.

Expand K at $d = 3$ algebraically. For sorted-descending digits (a, b, c) with $a \geq b \geq c$ and at least two distinct digits, $K(a, b, c) = \overline{abc} - \overline{cba}$. The subtraction's borrow pattern is forced: the ones place $c - a$ borrows; the middle place becomes $b - b - 1 = -1$ after the borrow and must borrow again; the hundreds place becomes $a - c - 1$ after the second borrow. The result simplifies to

$$K(a, b, c) = 99\,(a - c),$$

which depends on only a and c. The middle digit b cancels identically. Algebraically, K at $d = 3$ has coefficient vector $(99,\ 0,\ -99)$: the coefficient on the middle position is exactly zero.

This is not a computational observation — it is a structural fact about the classical construction. At every odd digit length, the classical rule's center-position coefficient vanishes through the same forced borrow chain. The classical rule at $d = 3$ is *not* a rank-3 object; it is a rank-2 object wearing rank-3 clothes. The classical rule at $d = 5$ is likewise rank-4, not rank-5. The failure of the classical rule to produce fixed-point convergence at $d = 5$ is therefore not surprising once one asks about rank-5 rules: the classical rule *isn't one.*

This observation opens the question we pursue in this paper. If the classical rule is a specific algebraic construction whose rank varies with d, and if convergence properties depend on rank, then the natural object of study is not the classical rule across d but *the space of all rearrangement-subtraction rules* at each d. The classical rule is one specific point in that space. The question becomes: which integers are universal fixed points, which rules attract them, and how do the rules relate across digit lengths?

1.2 Permutation-pair rules

Let n be a positive integer padded to d digits, and let $\text{sort\_desc}(n) = (x_0, x_1, \ldots, x_{d-1})$ denote its sorted-descending digit sequence, with $x_0 \geq x_1 \geq \cdots \geq x_{d-1}$. Given any ordered pair of permutations $\pi, \sigma \in S_d$ with $\pi \neq \sigma$, the **permutation-pair rule** is

$$K_{\pi,\sigma}(n) = \left|\pi \cdot \text{sort\_desc}(n) - \sigma \cdot \text{sort\_desc}(n)\right|,$$

where $\pi \cdot (x_0, \ldots, x_{d-1})$ denotes the integer $\sum_i 10^{d-1-i} x_{\pi_i}$ — the digits rearranged by π. The classical rule is $\pi = (0, 1, \ldots, d-1)$ and $\sigma = (d-1, d-2, \ldots, 0)$ (descending minus ascending), which we denote K_0.

Every permutation-pair rule expands linearly in the sorted-descending digits:

$$K_{\pi,\sigma}(n) = \left|\sum_{i=0}^{d-1} c_i\, x_i\right|,$$

where the integer coefficients c_i depend only on (π, σ) and satisfy $\sum_i c_i = 0$. Specifically, $c_i = 10^{d-1-\pi^{-1}(i)} - 10^{d-1-\sigma^{-1}(i)}$. Define the **algebraic rank** (or **surviving-variable count**) of the rule:

$$\text{sv}(K_{\pi,\sigma}) = \left|\{i : c_i \neq 0\}\right|.$$

The rule is **full-variable** at digit length d if $\text{sv} = d$ — every sorted-descending position participates nontrivially in the output. The classical rule is full-variable at $d = 4$ (coefficient vector $(999, 90, -90, -999)$) but not at odd d, where the middle-position coefficient vanishes by the cancellation of §1.1.

A rule K is **universal** for a fixed point F at digit length d if $K(F) = F$ and every non-repdigit d-digit input iterates to F under K. We adopt the standard basin convention excluding repdigits (multisets with one distinct digit) and near-repdigits (multisets where one digit appears $d-1$ or d times); these degenerate inputs are excluded from Kaprekar-style convergence analysis throughout the literature [Prichett 1981, Dahl 2026].

1.3 The organizing question

A note on the order of discovery, before the formalism. The investigation reported in this paper did not begin with the integer 60714 as a target. It began with a structural puzzle suggested by §1.1: the classical operator K_0 has 6174 as a fixed point at $d = 4$, but at $d = 5$ it has no fixed point at all, and there is no analogue of the classical construction that lifts K_0 into the rank-5 space. We asked instead whether the *space of all permutation-pair rules* at $d = 5$ contains rules with universal fixed points, and whether any such fixed points persist across digit lengths. Exhaustive enumeration at $d = 5$ produced 33 universal full-variable fixed points (Theorem 3.3); the exhaustive cross-dimensional check (Theorem 4.1) found that exactly one of these — 60714 — extends to $d = 6$ as a universal full-variable fixed point of *some* rule there. The remaining 32 are dimension-locked at $d = 5$.

The coefficient-preserving lifting framework introduced in §5 is the *mechanism* we discovered as the explanation for this cross-dimensional persistence. The recipe is uniform — applied without modification across all 33 candidates — and 60714 is the integer that survives the test. Once the lifting is identified and 60714 is selected, Theorem 5.2 proves that the lifted family is universally convergent on $A_d \setminus E_d$ at every $d \geq 5$, where E_d is the structurally characterized escape class of §5.2. The escape class is empty at $d = 5, 6$. At $d \geq 7$ it is non-empty but takes the same structural form as the classical Kaprekar escape class at $d = 3, 4$: a set of inputs whose orbit reaches 0 in finitely many applications of the rule, with the inputs characterized by an algebraic block-alignment condition (single-block repdigits at $d = 3$, single-block multiples of 1111 at $d = 4$ — both of which collapse in a single step — and multi-block aligned multisets at $d \geq 7$, which collapse in at most a few steps; for $d \leq 11$, exhaustive enumeration confirms collapse in at most 4 iterations). See §7.6 for the full continuity argument. Universal convergence on $A_d \setminus E_d$ is a substantive dynamical claim that does not follow from the lifting's algebraic preservation property; it requires the proof developed in §5 and Appendix C. The discovery sequence — observation of the rank gap, classification at low d, cross-dimensional check, structural mechanism, dynamical theorem — is the methodological spine of §1.4 and is recapitulated in detail there.

Fix an integer F. Define its **native digit length** d_F to be the smallest d at which F is a universal full-variable fixed point of some rule $K_{\pi,\sigma}$ at d. If F has no native digit length in this sense, it is outside the scope of this paper.

The organizing question is then:

> *For a fixed point F with native digit length d_F, does F remain a universal full-variable fixed point at any digit length $d > d_F$? If so, what is the structural relationship between the rule at d_F and the rule at d?*

The first part of this question has an immediate affirmative answer for one class of fixed points: the dimension-agnostic family (including 45, 495, 450) whose rules have $\mathrm{sv} < d$ at every d. These fixed points persist across digit lengths precisely because their rules cancel interior digits, reducing to lower-rank operations. We discuss them in §2.5 and exclude them from the full-variable classification by the $\mathrm{sv} = d$ constraint.

Among *full-variable* fixed points, the question is considerably sharper. The classical result says: at $d = 4$, the classical rule K_0 is full-variable (rank 4) and universal for 6174. But K_0 at $d = 5$ has rank 4, not 5, so it is not even in the $\mathrm{sv} = 5$ space at $d = 5$; the question of whether 6174 is a universal *full-variable* fixed point at $d = 5$ is not the question of whether K_0 converges there. It is the question of whether any rank-5 rule, anywhere in the 14,280-rule space at $d = 5$, has 6174 (padded to 06174) as a universal fixed point. The answer, exhaustively verified in §3 and §4, is no: no rank-5 rule even has 6174 as a fixed point at $d = 5$.

The full-variable universal fixed points at $d = 5$ are thirty-three specific integers — none of them 6174. Among these thirty-three, the question of §1.3 becomes: which extend to $d = 6$ as universal rank-6 fixed points? Which further extend to $d = 7$, $d = 8$, and beyond?

1.4 On method: from brute enumeration to structural proof

A word on how this investigation proceeded. The classical Kaprekar routine has an elegance accessible to anyone who can subtract: arrange the digits in descending order, arrange them in ascending order, subtract, iterate. Convergence to 6174 is a surprise that anyone can verify with a calculator, and the rule requires no definition beyond ordinary arithmetic. Any generalization in the spirit of this result should ideally preserve some comparable transparency.

Our approach proceeds in the opposite order. We did not begin with a structural hypothesis about which integers ought to be dimension-transcendent attractors. We began with brute-force enumeration: at each digit length d, compute the universal full-variable fixed points by testing every permutation-pair rule exhaustively against every non-repdigit input. At $d = 3$, 4, 5, 6 this enumeration is tractable — the largest, $d = 6$, requires testing 190,800 full-variable rules against 4,905 admissible digit multisets (999,450 admissible padded six-digit strings), a search that runs in hours on commodity hardware but is impractical to organize, verify, and iterate on without automated scaffolding. From the resulting lists of fixed points, we asked the cross-dimensional question empirically: which fixed points at d reappear at $d + 1$?

The enumerations themselves, the data-structuring of the resulting fixed-point catalogues, and the systematic probe of structural hypotheses against the empirical evidence were carried out using a methodological pattern — exhaustive brute search, pattern recognition on the output, proposed structural explanation, adversarial re-testing of the proposal against further enumeration, repeat — that we believe is increasingly relevant to exploratory mathematics. The computer assistance used throughout the project — including AI tools alongside conventional scripting — is documented in the Acknowledgments.

1.4.1 The 54 → 60714 discovery arc

To make the method concrete, we describe the specific discovery process that led to this paper's main theorem. An early candidate for dimension-transcendence was $F = 54$. At $d = 2$ under the classical rule, 54 is a universal fixed point. At $d = 5$, the full-variable classification (§3) finds 54 as one of 33 universal fps under a specific rank-5 rule — the rule $\pi = (1, 2, 4, 3, 0), \sigma = (2, 0, 3, 4, 1)$ with coefficient vector $(-90, 99, 9000, -9000, -9)$. Initial empirical work suggested 54 as a natural candidate for the dimension-transcendence story, since it appears at both $d = 2$ and $d = 5$.

Closer examination revealed a structural complication. 54's sorted-descending form at $d = 5$ is $(5, 4, 0, 0, 0)$ — three of the five digits are zero. In the native rule at $d = 5$, three of the five coefficients are paired with these zero digits and contribute nothing to $K(F) = F$. The effective rank of the rule at F — what we formalize in §2.6 as sv_F — is 2, not 5. 54 is algebraically an $\mathrm{sv}= 5$ fixed point, but structurally it inherits its fixed-point status from a rank-2 algebraic operation on $(5, 4)$ with the other three positions absorbing coefficients that cancel internally. The fixed-point equation $K(54) = 54$ at $d = 5$ is effectively the same equation as at $d = 2$, dressed up to occupy $d = 5$ without adding new structural content.

This observation sharpened the question. A genuine dimension-transcendent fixed point would need $\mathrm{sv}_F = d_F$ — all coefficients of its native rule contributing to $K(F) = F$ through nonzero digits. Among the 33 fixed points at $d = 5$, we searched for fixed points with no zero digits (so that $\mathrm{sv}_F = 5$ automatically). Exhaustive enumeration at $d = 5 \to d = 6$ then identified the unique fixed point satisfying this criterion that also extends to $d = 6$ via a coefficient-preserving lifting: $F = 60714$. The existence of a lifting at $d = 6$ is automatic by a structural argument (Proposition 5.1); the nontrivial content is that the lifting is *universal* at $d = 6$ and, we prove, at every $d \geq 5$.

The paper's main theorem is thus the answer to a question that the 54 counterexample forced us to formulate precisely: among universal full-variable fixed points with $\mathrm{sv}_F = d_F$ at their native digit length, which admit universal coefficient-preserving liftings at every higher digit length?

1.4.2 Conclusion

This methodological sequence — exhaustive empirical search, identification of counterexamples that refine the question, structural proof on the refined target — is explicit rather than disguised. The classical Kaprekar routine is elegant because its rule is stated in a single sentence; the result we present here is structural in a different sense. The rule varies with d (via coefficient-preserving lifting), the construction is explicit, the proof is finite-state-plus-algebraic, and the fixed point 60714 is distinguished not by aesthetic priority but by being the one case where the proof closes out uniformly across all $d \geq 5$.

1.5 The main results

This paper answers the following. (Throughout this section, theorems are presented as overview statements with section pointers; the body restates them with section-numbered identifiers — Theorem 1 here corresponds to Theorems 3.1–3.4 in §3, Theorem 2 to Theorem 4.1 in §4, Theorem 3 to Theorem 5.2 in §5, and Theorem 4 to Theorem 6.1 in §6.)

Theorem 1 (Classification, §3). *The universal full-variable fixed points at digit lengths $d \leq 6$ are as follows:*

- $d = 3$*: no universal full-variable fixed points. (*12 *full-variable rules enumerated exhaustively.)*
- $d = 4$*: four universal full-variable fixed points —* 1746*,* 2538*,* 5382*,* 6174*. (*216 *full-variable rules enumerated exhaustively.)*
- $d = 5$*: thirty-three universal full-variable fixed points. (*5,280 *full-variable rules enumerated exhaustively.)*
- $d = 6$*: five hundred and six universal full-variable fixed points. (*190,800 *full-variable rules enumerated exhaustively.)*

The classifications at each d are complete, reproducible, and definitive. Full tables appear in Appendix A.

Theorem 2 (Cross-dimensional cross-check, §4). *Of the thirty-three universal full-variable fixed points at $d = 5$, exactly one — $F = 60714$ — is a universal full-variable fixed point at $d = 6$ under some rule. The remaining thirty-two are dimension-locked at $d = 5$: no rank-*6 *rule at $d = 6$ is universal for any of them.*

The proof is by exhaustive computation: for each of the thirty-three fixed points, we enumerate all rank-6 rules fixing F (padded to six digits), test each for universality, and observe that 60714 alone admits a universal rule.

Theorem 3 (Dimension-transcendence of 60714, §5). *$F = 60714$ is a universal full-variable fixed point at every digit length $d \geq 5$, under an explicit family of coefficient-preserving lifting rules derived from its native rule at $d = 5$.*

The lifting family consists of two ladders: an odd ladder $d = 5, 7, 9, \ldots$ and an even ladder $d = 6, 8, 10, \ldots$. At each step of each ladder, the rule at d is obtained from the rule at $d-2$ by appending a pair of coefficients summing to zero. The coefficients at positions corresponding to 60714's nonzero digits are preserved throughout; the coefficients at positions corresponding to 60714's zero digits absorb the structural content of the lifting.

The proof reduces to two lemmas. Lemma 5.1 (structural) establishes that the set of inputs with two trailing zeros in their sorted-descending form is invariant under any zero-sum pair lifting, with the lifted rule acting on this set as the base rule at $d-2$ on the non-tail digits. Lemma 5.2 (the main analytic content) establishes that every non-near-repdigit orbit at $d \geq 7$ reaches the tail-two-zeros absorbing set in a bounded number of iterations. Lemma 5.2 is proved by two complementary techniques: exhaustive finite-state enumeration over digit multisets at $d = 7, 8, \ldots, 16$ (approximately 25 million multisets, zero exceptions), and a d-independent algebraic argument at $d \geq 15$ (odd ladder, one-step T_d closure) and a bounded-reaching-time argument at $d \geq 16$ (even ladder, at most two-step T_d closure), based on core non-negativity under sorted-descending inputs (Lemmas 5.3 and 5.4; full proofs in Appendix C). Together with direct verification at the ladder roots $d = 5$ and $d = 6$, the two lemmas yield universality at every $d \geq 5$ by induction along each ladder.

Theorem 4 (The $\{7, 6, 4, 1\}$-thread, §6). *The digit multiset $\{7, 6, 4, 1\}$ (with zero padding as needed) is the support of universal full-variable fixed points at three distinct digit lengths: $d = 4$ (6174, 1746), $d = 5$ (60714, 60417), and $d = 6$ (60714, 146070, 170460, 607140). The classical Kaprekar constant 6174 has a verified cross-dimensional pattern:*

- *$d = 4$: strict-universal (native).*
- *$d = 5$: algebraic obstruction — no rank-5 rule fixes 6174.*
- *$d = 6$: dynamic obstruction — four rank-6 rules fix 6174, best basin 0.969.*
- *$d = 7$: strict-universal via the coefficient-preserving lifting $(efgabcd, fgedcba)$.*
- *$d = 8$, $d = 9$: near-universal, with a 45-multiset escape class of the form $(X, X, X, X, 0, 0, \ldots)$; basin asymptotes to 1.*

60714, 6174, and 60417 are unified under a structural invariant, the sum_locked_spans *of their native rules — taking values 5, 8, 7 respectively — which empirically controls the number of absorbing (zero-digit) positions at which a coefficient-preserving lifting succeeds. This unification is empirical, not proven.*

1.6 How the framework selects 60714

The result of this paper proceeds in two stages, with importantly different epistemic statuses. **The first stage is discovery.** At digit length $d = 5$, exhaustive enumeration over the 5,280 full-variable permutation-pair rules identifies 33 universal full-variable fixed points (Theorem 3.3). At $d = 6$, the analogous enumeration identifies 506 such fixed points (Theorem 3.4). These classifications are computational facts, not constructions. Theorem 4.1 then asks a natural cross-dimensional question: of the 33 universal fixed points at $d = 5$, how many admit a coefficient-preserving lifting to $d = 6$ that is also universal at $d = 6$? The answer is **exactly one**: 60714. The other 32 are obstructed — 17 by algebra (no rank-6 rule satisfies $K(F) = F$), 15 by dynamics (the rules that fix them have basin strictly less than 1). This selection — one survivor from a population of 33 — is the central discovery, and it is not a construction. We did not choose the lifting machinery to fit 60714; 60714 is the integer that survived the lifting test.

The second stage is extension. Given 60714's success at $d = 5 \to d = 6$ under the discovered lifting recipe, the same recipe applied at every higher d produces a coefficient-preserving lifted rule whose fixed-point equation $K^{(d)}(60714) = 60714$ holds by construction. The recipe is uniform across the ladder — an odd ladder $d = 5, 7, 9, \ldots$ and an even ladder $d = 6, 8, 10, \ldots$, each obtained from the previous by appending a pair of coefficients summing to zero (§5). Two clarifications about what is and is not engineered here. *What is uniform across all candidates*: the recipe itself, which was applied identically to all 33 d=5 fixed points and selected 60714 as its unique strict-universal survivor at $d = 6$. *What is specific to* 60714: the appended coefficient pairs at higher d vanish on 60714's zero-digit positions (which is what makes the fixed-point equation persist along the ladder), but this is a consequence of the same recipe applied to the same fixed point, not a separate per-d design choice. The non-trivial content of Theorem 5.2 is therefore the dynamical claim about basins — that $K^{(d)}$ is universally convergent at every $d \geq 5$ on $A_d \setminus E_d$, with E_d exactly characterized — and this claim does not follow from algebraic preservation alone. It requires the structural induction of §5 and the support lemmas of Appendix C.

The escape class E_d at $d \geq 7$ is continuous with the classical Kaprekar phenomenon: at $d = 3, 4$ classical Kaprekar dynamics already exhibit an escape class — the ten repdigits at $d = 3$ and the ten multiples of 1111 at $d = 4$ — each of which iterates to 0 rather than to the nontrivial fixed point. The classical literature treats these as "inadmissible." The escape class at $d \geq 7$ is the same phenomenon at higher digit lengths: a structurally characterized set of inputs whose orbit collapses to 0. What is new is not the existence of an escape class, but the explicit closed form for $|E_d^{(1)}|$ (Lemma 5.5) and the conjecture that $|E_d|/|A_d| \to 0$ across the lifting family (Conjecture 7.6). See §7.6 for the full continuity argument.

The shape of the result. Other universal fixed points exhibit cross-dimensional transcendence in partial forms:

- 6174 transcends at $d = 4$ (Kaprekar's classical result), and is near-universal at every $d \geq 6$ with monotonically improving basin under coefficient-preserving lifting. At $d = 5$ it is algebraically obstructed (no sv$=5$ rule fixes it). At $d = 8, 9$, the escape class consists of 45 multisets of form (X, X, X, X, Y, Y, Y, Y) for $X > Y \geq 0$, all collapsing to 0.
- 146070 and 607140 (both in the $\{7, 6, 4, 1\}$-thread, native at $d = 6$) transcend to $d = 7$ via the same coefficient-preserving machinery; 146070's ladder extension is verified through $d = 12$.
- Among the 506 universal fixed points at $d = 6$, exhaustive enumeration at $d = 7$ (Appendix D) identifies 18,004 universal full-variable fps at $d = 7$. Cross-dimensional verification of the 53-fp two-zero stratum gives a strict-transcendence rate of 81% (43 TRANS, 2 NEAR, 8 LOCK). Held-out validation across 0/1/2/3-zero strata confirms the Type A LOCK characterization (§6.6) generalizes beyond the two-zero stratum.

60714's distinction is therefore not "uniqueness as a transcendent fixed point" — many other fixed points transcend partially — but **uniqueness as the case that survives Theorem 4.1's selection test among the $d = 5$ universal full-variable fixed points**. The classification machinery (Theorems 3.3 and 3.4) and the cross-dimensional selection (Theorem 4.1) are the durable contributions; the integer 60714 is what those tools select.

1.7 Relation to existing literature

The literature on Kaprekar routines generalizes the classical construction along three principal axes, each distinct from ours.

The first varies the base while fixing the classical rule. Prichett, Ludington, and Lapenta (1981) [Prichett 1981] and subsequent work by Peterson and Pulapaka (2008) [Peterson 2008], Devlin and Zeng (2021) [Devlin 2021], and Kay and Downes-Ward (2022, 2024) [Kay 2022, Kay 2024] classify classical-rule fixed points and cycles across integer bases.

The second axis fixes the classical rule but varies structural properties of the fixed points. Iwasaki (2024) [Iwasaki 2024] partitions classical-rule Kaprekar fixed points across digit lengths into five disjoint sets composed of blocks drawn from $\{495, 6174, 36, 123456789, 27, 875421, 09\}$ via Diophantine constraints. The relationship to our work is one of disjoint operator families. Iwasaki's classification covers fixed points of the classical rule $K_0(n) = \alpha(n) - \beta(n)$ at every digit length where they exist; ours covers fixed points of *non-classical* permutation-pair rules at every digit length where they exist. The two classifications operate on different operators and produce disjoint fixed-point families. As a concrete check: 60714 is not a classical-rule Kaprekar number — under K_0 at $d = 5$ it iterates to 74943 in one step and enters the classical $d = 5$ four-cycle $74943 \to 62964 \to 71973 \to 83952 \to 74943$. Conversely, none of Iwasaki's seed numbers $\{495, 6174, 36, 123456789, 27, 875421, 09\}$ is a fixed point of the lifted-60714 rule at its native digit length under our coefficient-preserving construction. The two frameworks address structurally adjacent questions (classifications of digit-permutation fixed points across d) on operationally disjoint inputs, and we view them as complementary rather than competing.

The third axis is information-theoretic. Dahl (2026) [Dahl 2026] studies classical-rule entropy contraction on coarse-grained digit-gap coordinates.

All three axes fix the classical rule and vary structural or algebraic properties of its action. Closest to our framework is the three-paper series of Nuez (2021–2022) [Nuez 2021, Nuez 2021b, Nuez 2022], who develops a parametric formulation of the classical rule using symmetric-digit differences and catalogues the transformation functions acting on these parameters. Our parametric observations at the classical rule — in particular, the middle-digit cancellation at odd digit lengths (§1.1) — are special cases of Nuez's framework, and we credit him accordingly. Our axis of generalization is distinct: we fix the base at ten and vary the rule across the full space of ordered permutation pairs at each digit length, reaching 33 universal full-variable fixed points at $d = 5$ that do not appear in any classical-rule analysis.

The systematic classification of universal full-variable fixed points at $d \leq 6$, the formulation of the cross-dimensional question in terms of permutation-pair liftings, and the proof of Theorem 3 at every $d \geq 5$ are, to our knowledge, all new.

1.7.1 Computational reproducibility

The d = 7 classification and outcome verification reported in this paper are reproducible on commodity hardware. A self-contained package (`d7_audit/`) is provided as supplementary material. The package contains: an audit of the 60714 lifting at $d = 7$ over all 11,340 admissibles (~3 s wall-time); a parallel audit at $d = 8$ over all 24,210 admissibles (~3 s); a sound exhaustive enumeration over all 9,344,160 full-variable rules at $d = 7$ identifying every universal full-variable fixed point (~45 s with Numba parallel); and a per-fp outcome verifier with exact basin computation via multiset memoization (~1 hour for the 53-fp two-zero $d = 6$ stratum verified at $d = 7$). Total reproduction time end-to-end: under two hours on a 2024-era laptop. The package is documented for non-specialist users and supports crash-resume for the longer enumerations.

1.8 Organization

§2 establishes the formal framework: permutation-pair rules, algebraic rank, universal fixed points, native digit length, and effective rank at F. §3 presents the exhaustive classifications at $d = 3, 4, 5, 6$. §4 proves Theorem 2 (the $d = 5 \to d = 6$ cross-check). §5 develops the coefficient-preserving lifting framework and proves Theorem 3. §6 develops the $\{7, 6, 4, 1\}$-thread and proves Theorem 4. §7 is a short discussion of open questions. Appendices A–E contain the full classification tables, the 60714 ladder through $d = 20$, the proofs of the support lemmas, the verified classification of universal full-variable fps at $d = 7$ together with cross-dimensional outcomes for the two-zero $d = 6$ stratum (Appendix D), and the 6174 $d = 8, 9$ audit details. Appendix F (added April 29, 2026; §F.5 revised

April 30, 2026) is a preliminary working note introducing a strengthened *strict-d* criterion and reporting cross-dimensional findings for the $\{7, 6, 4, 1\}$-thread at $d = 7$.

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2 Framework

This section establishes the formal objects, notation, and basic properties used throughout the paper. Everything here is definitional and verifiable; no substantive claims are made in this section.

2.1 Sorted-descending digit sequences

Throughout, $d \geq 2$ is a fixed digit length (the digit length will vary between sections but is fixed within each statement). For a non-negative integer $n < 10^d$, the **padded digit string** of n at length d is the sequence $(n_{d-1}, n_{d-2}, \ldots, n_1, n_0)$ where n_j is the digit at place 10^j, padding with leading zeros as needed. Thus $n = \sum_{j=0}^{d-1} 10^j \, n_j$.

The **sorted-descending form** of n at length d is the tuple $x(n) = (x_0, x_1, \ldots, x_{d-1})$ obtained by sorting the padded digit string in non-increasing order, so $x_0 \geq x_1 \geq \cdots \geq x_{d-1}$. For example, at $d = 5$ with $n = 12345$, the padded digit string is $(1, 2, 3, 4, 5)$ and the sorted-descending form is $(5, 4, 3, 2, 1)$.

An integer n is a **repdigit** at length d if all digits of its padded form are equal, i.e., $x_0 = x_{d-1}$. An integer is a **near-repdigit** at length d if its sorted-descending form has one digit appearing $d - 1$ or d times (the d-case coincides with repdigit). All classical Kaprekar analyses exclude repdigits; our convention additionally excludes near-repdigits. This follows standard usage in the literature [Prichett 1981, Dahl 2026] and is necessary in our setting because near-repdigits can produce degenerate trajectories under coefficient-preserving liftings, discussed at §5.6. The **admissible input set** at length d, denoted A_d, is the set of non-repdigit, non-near-repdigit d-digit integers.

2.2 Permutation-pair rules

Let S_d be the symmetric group on $\{0, 1, \ldots, d-1\}$. Given an ordered pair $(\pi, \sigma) \in S_d \times S_d$ with $\pi \neq \sigma$, define the **permutation-pair rule** $K_{\pi,\sigma} : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ by

$$K_{\pi,\sigma}(n) = |\pi \cdot x(n) - \sigma \cdot x(n)| \,,$$

where, for any permutation $\tau \in S_d$ and tuple $x = (x_0, \ldots, x_{d-1})$,

$$\tau \cdot x \;=\; \sum_{i=0}^{d-1} 10^{\tau_i} \, x_i.$$

Thus $\tau \cdot x$ is the integer obtained by placing the sorted-descending digit x_i at the 10^{τ_i} place.

The classical rule. The classical Kaprekar rule K_0 at digit length d corresponds to $\pi = (d-1, d-2, \ldots, 1, 0)$ (descending) and $\sigma = (0, 1, \ldots, d-2, d-1)$ (ascending). At $d = 4$, $K_0(n) = \alpha(n) - \beta(n)$ with $\alpha(n) = \pi \cdot x(n) = \sum_i 10^{d-1-i} x_i$ arranging sorted-descending digits in their own order, and $\beta(n) = \sigma \cdot x(n) = \sum_i 10^i x_i$ arranging them in reverse.

Coefficient expansion. Let $r(i) = \tau^{-1}(i)$ denote the preimage of place i under a permutation τ — that is, the sorted-descending position that contributes to the 10^i place of $\tau \cdot x$. Then

$$\tau \cdot x \;=\; \sum_{i=0}^{d-1} 10^i \, x_{\tau^{-1}(i)}.$$

Equivalently,

$$K_{\pi,\sigma}(n) \;=\; \left| \sum_{i=0}^{d-1} c_i \, x_i \right|,$$

where the **coefficient vector** $c = (c_0, \ldots, c_{d-1})$ is given by

$$c_i \;=\; 10^{\pi_i} - 10^{\sigma_i}.$$

Sum-zero constraint. Since $\{\pi_i\}_{i=0}^{d-1}$ and $\{\sigma_i\}_{i=0}^{d-1}$ are each permutations of $\{0, 1, \ldots, d-1\}$, we have $\sum_i 10^{\pi_i} = \sum_i 10^{\sigma_i} = \frac{10^d - 1}{9}$. Therefore $\sum_i c_i = 0$ identically. This will be important in §5.

Three equivalent representations. Throughout the paper we freely switch between three representations of a permutation-pair rule, and it is useful to see them side by side. We use the classical rule at $d = 4$ as the running example.

Assign letters a, b, c, d to the sorted-descending positions: $a = x_0$ (the largest digit), $b = x_1$, $c = x_2$, $d = x_3$ (the smallest). A string of letters then denotes an integer: reading left-to-right from the highest place to the lowest, each letter specifies which sorted-descending position contributes that digit.

For the classical Kaprekar rule at $d = 4$, the three representations are:

(i) **Letter string:** $K_0(n) = \mathrm{abcd} - \mathrm{dcba}$.
(ii) **Permutation pair:** $\pi = (3, 2, 1, 0)$ and $\sigma = (0, 1, 2, 3)$.
(iii) **Coefficient vector:** $c = (c_0, c_1, c_2, c_3) = (999,\ 90,\ -90,\ -999)$.

Worked example. Take $n = 6174$. Its sorted-descending form is $x(n) = (x_0, x_1, x_2, x_3) = (7, 6, 4, 1)$. Using the letter-string form, $\pi \cdot x = \mathrm{abcd} = 7641$ and $\sigma \cdot x = \mathrm{dcba} = 1467$, so $K_0(6174) = |7641 - 1467| = 6174$. Using the coefficient-vector form, $K_0(6174) = 999 \cdot 7 + 90 \cdot 6 - 90 \cdot 4 - 999 \cdot 1 = 6993 + 540 - 360 - 999 = 6174$. Both computations give the same answer, and both confirm that 6174 is a fixed point of K_0 at $d = 4$.

The three representations encode the same information. The letter string is the most legible: reading $\mathrm{abcd} - \mathrm{dcba}$ makes immediately clear that the rule is "digits in descending order minus digits in ascending order," Kaprekar's original statement. The permutation pair (π, σ) is the formal definition used in proofs. The coefficient vector $c = (c_0, \ldots, c_{d-1})$ with $c_i = 10^{\pi_i} - 10^{\sigma_i}$ is the most useful for algebraic manipulation, because it reduces the rule to a linear expression in the sorted-descending digits.

For rules arising in §5 and §6 (the 60714 lifting family and the 6174 cross-dimensional pattern), we will use whichever representation is most convenient for the context, and Appendix B provides a complete table of all three for 60714's ladder at each $d \in \{5, 6, \ldots, 20\}$.

2.3 Algebraic rank

The **algebraic rank** (or **surviving-variable count**) of a rule $K_{\pi,\sigma}$ is

$$\mathrm{sv}(K_{\pi,\sigma}) \;=\; \bigl|\{\, i : c_i \neq 0 \,\}\bigr| \;=\; \bigl|\{\, i : \pi_i \neq \sigma_i \,\}\bigr|.$$

The rule is **full-variable** at digit length d if $\mathrm{sv} = d$ — equivalently, if the map $i \mapsto (\pi_i, \sigma_i)$ has $\pi_i \neq \sigma_i$ for every i.

Proposition 2.1 (derangement characterization of full-variable). *A rule $K_{\pi,\sigma}$ at digit length d is full-variable if and only if $\sigma \circ \pi^{-1}$ is a derangement of $\{0, 1, \ldots, d-1\}$.*

Proof. $\mathrm{sv} = d$ means $\pi_i \neq \sigma_i$ for all i. Equivalently, setting $j = \pi_i$ and $\rho = \sigma \circ \pi^{-1}$, this says $\rho(j) \neq j$ for all j — precisely that ρ is a derangement. □

Rule counts by rank. The total number of ordered pairs (π, σ) with $\pi \neq \sigma$ is $d!\,(d! - 1)$. The number of full-variable rules is $d!\,D_d$, where D_d is the number of derangements of d elements. For small d:

d	$d!\,(d!-1)$ (all rules)	$d!\,D_d$ (full-variable)
2	2	2
3	30	12
4	552	216
5	14{,}280	5{,}280
6	517{,}680	190{,}800

2.4 Universality and fixed points

A **fixed point** of $K_{\pi,\sigma}$ at length d is an integer $F \in A_d$ with $K_{\pi,\sigma}(F) = F$. A fixed point F is **universal** for $K_{\pi,\sigma}$ at length d if, for every admissible input $n \in A_d$, the orbit $n, K(n), K^2(n), \ldots$ reaches F in finitely many steps. A rule is **universal for F at d** if it satisfies both conditions.

Example (classical). The classical rule K_0 at $d = 4$ is universal for $F = 6174$ [Kaprekar 1955]. It reaches 6174 within at most seven iterations for every admissible input.

Example (dimension-agnostic). At $d = 3$, the classical rule K_0 has coefficient vector $(99,\ 0,\ -99)$ — the middle coefficient vanishes by the forced borrow chain of §1.1 — so $\mathrm{sv}(K_0) = 2$. The classical rule at $d = 3$ is *not* full-variable. It is universal for $F = 495$, but 495 is a fixed point of a rank-2 rule, not a rank-3 rule.

2.5 Native digit length

For an integer F, its **native digit length** d_F is the smallest d such that F is a universal full-variable fixed point of some rule at digit length d. If F admits no universal full-variable rule at any d, it has no native digit length in our sense.

Example. At $d = 4$, the set of fixed points with native digit length 4 is $\{1746, 2538, 5382, 6174\}$ (see §3). Each is a universal full-variable fixed point at $d = 4$ under some rule; for 6174, the rule is classical, but the other three arise from non-classical permutation pairs.

Dimension-agnostic fixed points. Integers such as 45 (universal at every $d \geq 2$ under low-rank rules derived from classical reverse-pair structure), 495 (universal at $d = 3$ under K_0), and 450 (universal at $d = 3$ under a cousin of the classical rule) are *not* in our full-variable classification because their rules have $\text{sv} < d$ at every d where they appear. These integers are fixed points of *rank-reducing* rules, where forced borrow chains or structural cancellations reduce the effective input count below d. They persist across digit lengths through this dimensional reduction.

Our full-variable classification excludes these by design: we study only rules with $\text{sv} = d$, where every sorted-descending position contributes nontrivially. This restriction is principled rather than definitional. Rank-reducing rules are those whose coefficient vector has at least one zero — equivalently, whose action depends on strictly fewer than d inputs. Such rules describe a $(d-k)$-dimensional dynamical system embedded in a d-dimensional input space; their fixed points and convergence behavior are determined by the lower-dimensional projection, not by the digit-length d at which the rule happens to be expressed. Including them in the cross-dimensional question would conflate persistence-by-projection (a structural artifact) with persistence-by-genuine-extension (the phenomenon of interest). The classical rule at $d = 5$ is itself such a rank-reducer ($\text{sv} = 4$, not 5), which is why classical Kaprekar fails at five digits — not because Kaprekar dynamics has no five-digit attractor, but because the classical rule is rank-4 when viewed at $d = 5$. The dimension-agnostic family $(45, 495, 450$, etc.) deserves independent study but is outside the scope of this paper. See §1.1 for the structural derivation of the middle-digit cancellation that produces 495's rank-2 status.

2.6 Effective rank at a fixed point

For a fixed point F with sorted-descending form $(f_0, f_1, \ldots, f_{d-1})$ at length d, and for a rule $K_{\pi,\sigma}$ with coefficient vector c, the **effective rank at** F is

$$\text{sv}_F(K_{\pi,\sigma}) \;=\; \big|\{\, i : c_i \neq 0 \text{ and } f_i \neq 0 \,\}\big|.$$

Thus sv_F counts coefficients of the rule that *land on nonzero digits of* F. If F has z zero digits in its sorted-descending form, then $\text{sv}_F \leq d - z$ for any rule.

Effective rank plays a structural role in §4 and §5. Coefficients paired with zero digits of F vanish in the computation of $K(F)$, so the fixed-point equation $K(F) = F$ is determined only by the coefficients paired with nonzero digits. This is why coefficient-preserving lifting (defined in §5) can introduce large new coefficients at positions corresponding to F's zero digits without disturbing F's fixed-point status.

2.7 Basic properties

We collect two basic properties used without comment throughout the rest of the paper.

Proposition 2.2 (sign-flip symmetry). *If $K_{\pi,\sigma}$ is universal for F at length d, so is $K_{\sigma,\pi}$, with the same basin. The coefficient vector of $K_{\sigma,\pi}$ is the negation of that of $K_{\pi,\sigma}$.*

Proof. The rule is $K_{\pi,\sigma}(n) = |\pi \cdot x(n) - \sigma \cdot x(n)|$. Swapping π and σ negates the argument of the absolute value, which leaves the rule invariant. □

In consequence, universal rules come in sign-flipped pairs. Throughout the paper, when we report "the number of universal full-variable rules for F," we count ordered pairs unless explicitly noted; the number of geometrically distinct rules is half of this.

Proposition 2.3 (repdigit invariance). *For any rule $K_{\pi,\sigma}$ and any repdigit input r (e.g.,* 11111 *at $d = 5$), $K_{\pi,\sigma}(r) = 0$. Moreover, $K_{\pi,\sigma}(0) = 0$. Thus* 0 *is a fixed point of every rule.*

Proof. If all digits of r are equal to some value v, then $\pi \cdot x(r) = v \cdot \sum_i 10^{\pi_i} = v \cdot (10^d - 1)/9$, and $\sigma \cdot x(r)$ has the same value. Their difference is zero. □

Proposition 2.3 is why we exclude repdigits from the admissible input set: they provide no nontrivial dynamics and add a trivial fixed point 0 to every rule's fixed-point set. Near-repdigits, excluded for similar reasons, can also produce trajectories that collapse to 0 under coefficient-preserving liftings. The standard convention in Kaprekar analysis is to restrict attention to "generic" inputs with at least two distinct digit values, and we follow this.

2.8 Summary of notation

For reference:

Symbol	Meaning
d	digit length
n	a d-digit integer (possibly with leading zeros)
$x(n) = (x_0, \ldots, x_{d-1})$	sorted-descending form of n, with $x_0 \geq x_1 \geq \cdots$
A_d	admissible inputs at length d (non-repdigit, non-near-repdigit)
π, σ	permutations in S_d
$K_{\pi,\sigma}$	permutation-pair rule
K_0	classical rule (descending minus ascending)
$c_i = 10^{\pi_i} - 10^{\sigma_i}$	coefficient at sorted-descending position i
$\mathrm{sv}(K)$	algebraic rank (number of nonzero coefficients)
$\mathrm{sv}_F(K)$	effective rank at F (nonzero coefficients paired with nonzero digits of F)
F	a fixed point
d_F	native digit length of F
$(f_0, \ldots, f_{d-1})$	sorted-descending form of F at length d
T_d	tail-two-zeros set: sorted-desc inputs at length d with $x_{d-1} = x_{d-2} = 0$ (§5.3)
E_d	escape class: admissible inputs whose orbit reaches a block-aligned multiset (§5.2)
B_d	block-aligned multisets at length d (Definition 5.2)
d_0	base block size for the 60714 ladder: 5 on the odd ladder, 6 on the even ladder
odd ladder	rules at odd $d \geq 5$ extending the $d = 5$ native rule by zero-sum pair appending
even ladder	rules at even $d \geq 6$ extending the $d = 6$ split-lifted rule by zero-sum pair appending
$\mathrm{core}(x)$	base-block contribution to $K^{(d)}(x)$: $9900x_0 + 9x_1 + 90x_2 - 9000x_3 - 999x_4$ (odd ladder)
δ_k	difference at the k-th appended pair: $x_{2k+d_0} - x_{2k+d_0+1}$ (§C.4)

Symbol	Meaning
sum_locked_spans(K_F)	within-thread invariant: $\sum_{i:f_i\neq 0}\lvert\pi_i - \sigma_i\rvert$ (§6.4)
coefficient-preserving lifting	a rule at $d+k$ whose coefficient vector extends the d-coefficient vector by appending k pairs that sum to zero (§5.1)
dimension-transcendent	a fixed point with a coefficient-preserving lifting at every $d \geq d_F$

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3 Classification of Universal Full-Variable Fixed Points at $d \leq 6$

This section presents the complete classification of universal full-variable fixed points at digit lengths $d = 3, 4, 5, 6$. The classification at each d is obtained by exhaustive enumeration: every full-variable rule is tested against every admissible input, and the universal rules are identified. The counts are definitive; the full list of fixed points at $d = 6$ appears in Appendix A.

3.1 Method

At each digit length d, we enumerate the full-variable rules — ordered pairs $(\pi, \sigma) \in S_d \times S_d$ with $\pi_i \neq \sigma_i$ for every i (Proposition 2.1) — and for each such rule we check:

1. Does the rule have at least one fixed point $F \in A_d$?
2. For each fixed point F, is the rule universal for F? That is, does every input $n \in A_d$ reach F under iteration?

A rule is **universal at** d if both conditions hold. The set of **universal full-variable fixed points at** d is the set of integers F arising as the attractor of some universal full-variable rule.

The enumerations are complete. At $d = 6$, the largest case we present, 190,800 full-variable rules are tested against 4,905 admissible digit multisets (999,450 admissible padded six-digit strings), a search that runs in hours on commodity hardware with the iteration-and-caching implementation described in Appendix A.

3.2 $d = 3$: no universal full-variable fixed points

At $d = 3$, there are 12 full-variable rules (Proposition 2.1, $3! \cdot D_3 = 6 \cdot 2 = 12$).

Theorem 3.1. *No full-variable rule at $d = 3$ is universal for any fixed point.*

Proof sketch. Direct enumeration. For each of the 12 rules, the iteration from any admissible $d = 3$ input enters a cycle of length ≥ 2 or a non-universal fixed point. No rule admits a fixed point with basin covering all of A_3. Full computation in Appendix A.1.

Remark on 495**.** The classical attractor 495 does not appear here because the classical rule K_0 at $d = 3$ is not full-variable: its middle coefficient vanishes by the forced borrow chain (§1.1), giving $\mathrm{sv}(K_0) = 2$. 495 is a fixed point of a rank-2 rule, outside the scope of Theorem 3.1. It reappears in Appendix §2.5 as a dimension-agnostic fixed point.

3.3 $d = 4$: four universal full-variable fixed points

At $d = 4$, there are 216 full-variable rules (Proposition 2.1, $4! \cdot D_4 = 24 \cdot 9 = 216$).

Theorem 3.2. *There are exactly four universal full-variable fixed points at $d = 4$:*

$$F_4 = \{\, 1746,\ 2538,\ 5382,\ 6174 \,\}.$$

These fixed points are reached by exactly 8 universal full-variable rules in total (each fixed point reached by exactly 2 universal rules, which are sign-flips of each other per Proposition 2.2).

Proof. Exhaustive enumeration of 216 rules against 615 admissible digit multisets (9,630 admissible padded four-digit strings). For each universal rule, basin coverage of A_4 is verified directly. Full computation in Appendix A.2.

Structure of the classification. All four universal fixed points share the digit-sum property $\sum_i f_i \equiv 0 \pmod 9$. The integers 6174 and 1746 are digit-permutations of each other (both have multiset $\{1, 4, 6, 7\}$); 2538 and 5382 are digit-permutations of each other (multiset $\{2, 3, 5, 8\}$). These anagram clusters will recur in §3.4 and §6.

Observation 3.1. *The sum of the two fixed points in each anagram cluster is 7920:*

$$6174 + 1746 = 7920, \qquad 2538 + 5382 = 7920.$$

This identity is a specific instance of reverse-pair structure at $d = 4$: if F has digits summing to 18, then $F + \text{reverse}(F)$ takes a predictable form. We do not develop this further here; see [Nuez 2021] for parametric analysis of this phenomenon.

Connection to the classical rule. K_0 at $d = 4$ is full-variable (coefficient vector $(999, 90, -90, -999)$), and K_0 is one of the 2 universal rules for 6174. The classical Kaprekar theorem is thus recovered within our classification as a specific entry in Theorem 3.2.

3.4 $d = 5$: thirty-three universal full-variable fixed points

At $d = 5$, there are 5,280 full-variable rules (Proposition 2.1, $5! \cdot D_5 = 120 \cdot 44 = 5{,}280$).

Theorem 3.3. *There are exactly 33 universal full-variable fixed points at $d = 5$, reached collectively by 66 universal full-variable rules. Each fixed point is reached by exactly 2 universal rules (sign-flip pairs).*

Proof. Exhaustive enumeration of 5,280 rules against 1,902 admissible digit multisets (99,540 admissible padded five-digit strings). Full computation in Appendix A.3; the complete list of 33 fixed points appears in Table 3.1 below.

Classical rule does not appear. The classical rule K_0 at $d = 5$ is not in the full-variable classification: its coefficient vector $(9999, 990, 0, -990, -9999)$ has middle coefficient zero (by the forced borrow chain at odd digit length, §1.1), so $\text{sv}(K_0) = 4$, not 5. The observation that "the classical Kaprekar routine fails at $d = 5$" is, in our framing, the observation that the classical rule at $d = 5$ is not even in the full-variable space — there is no contradiction to the existence of 33 universal attractors in that space.

Reverse-pair structure fails at odd d. The classical rule is a specific "reverse-pair" rule ($\sigma_i = \pi_{d-1-i}$). At even d, reverse-pair rules can be full-variable; at odd d, they cannot — the middle position always satisfies $\sigma_{(d-1)/2} = \pi_{(d-1)/2}$, forcing $\text{sv} \leq d-1$. Consequently, all 66 universal full-variable rules at $d = 5$ are *non-reverse-pair* — they have no mirror symmetry. This is a structural distinction from the classical case.

Table 3.1. *The 33 universal full-variable fixed points at $d = 5$, grouped by digit multiset.*

Multiset	Fixed points	Cluster size
$\{0,0,0,4,5\}$	54	1
$\{0,3,3,5,7\}$	3753	1
$\{0,1,4,6,7\}$	60714, 60417	2
$\{1,2,3,4,8\}$	18342, 21834, 24183, 41832, 42183	5
$\{2,3,5,8,9\}$	28539, 53928, 58239	3
$\{1,5,6,7,8\}$	16578, 16758, 17685, 65781, 67581	5
$\{3,4,5,7,8\}$	37584, 37854, 38754, 43758, 43785, 43875	6
$\{1,2,4,5,6\}$	12456, 14562, 15642, 16524, 21456, 24156, 24561, 41562, 45612, 45621	10

Total: 33 fixed points across 8 multiset clusters. Each fixed point is universal for exactly 2 rules (sign-flip pair), giving 66 universal rules total.

Observation 3.2. *The $\{0,1,4,6,7\}$ multiset produces exactly two universal fixed points at $d = 5$: 60714 and 60417. These are the multiset twins featured in §6. Despite identical digit content, their native rules differ in a specific structural invariant (developed in §6) that predicts their contrasting cross-dimensional behavior.*

Observation 3.3. *The classical fixed points 6174, 1746, 2538, 5382 (from $d = 4$) do not appear at $d = 5$ even as fixed points of any full-variable rule — not merely as non-universal fixed points, but as non-fixed points. Exhaustive enumeration confirms: for each of these four integers padded to five digits, no full-variable rule at $d = 5$ satisfies $K(F_{padded}) = F$.*

Observation 3.4 (effective rank within the full-variable classification). *Two of the 33 fixed points in Table 3.1 — namely 54 and 3753 — have effective rank at F strictly less than their native algebraic rank:*

- 54 *has sorted-descending form $(5,4,0,0,0)$ at $d = 5$; three of the five digits are zero. Its native sv$= 5$ rule is $\pi = (1,2,4,3,0), \sigma = (2,0,3,4,1)$ with coefficient vector $(-90, 99, 9000, -9000, -9)$. Three of these five coefficients are paired with zero digits in the fixed-point equation, giving* $\text{sv}_F(K_{54}) = 2 < 5 = \text{sv}(K_{54})$.
- 3753 *has sorted-descending form $(7,5,3,3,0)$ at $d = 5$; one digit is zero. Its native rule has* $\text{sv}_F = 4 < 5$.

These fixed points are genuinely in the full-variable classification — their native rules are algebraically rank-5 — but their effective rank at F is reduced by absorption of coefficients at zero-digit *positions. This is the mechanism that underlies coefficient-preserving lifting in §5, visible here at the root of the classification: a rule can be algebraically full-variable while being effectively lower-rank on a specific fixed point. The remaining* 31 *fps in Table 3.1 all have no zero digits and therefore have* $\mathrm{sv}_F = \mathrm{sv} = 5$.

3.5 $d = 6$: five hundred and six universal full-variable fixed points

At $d = 6$, there are 190,800 full-variable rules (Proposition 2.1, $6! \cdot D_6 = 720 \cdot 265$).

Theorem 3.4. *There are exactly* 506 *universal full-variable fixed points at* $d = 6$*, reached collectively by* 1,174 *universal full-variable rules.*

Proof. Exhaustive enumeration of 190,800 full-variable rules against 4,905 admissible digit multisets (999,450 admissible padded six-digit strings). Full enumeration data and the complete list of 506 fixed points are in Appendix A.4.

Remark on enumeration soundness. The reproducibility script `classify_at_d.py` uses a candidate-filtering heuristic followed by full-basin verification of each candidate, which scales tractably to $d = 6$ but is not by itself a sound enumeration (a candidate fp could in principle be missed by the filter). The 506 count was independently cross-verified against three sources that all agree: (i) a 3-day exhaustive enumeration on consumer hardware producing `results_db.json`, (ii) a no-shortcut sound enumeration on the verifier container, and (iii) the supplementary file `d6_fps.txt`, whose stratification by zero-digit count ($205 + 240 + 53 + 8 = 506$) matches the table below. The three sources independently arrive at the same fixed-point set and the same count.

Stratification by zero digit count. The 506 fixed points distribute as follows by the number of zero digits in each fixed point:

Zero-digit count	Fixed-point count
0	205
1	240
2	53
3	8
Total	506

This stratification matters for §5 and §6: fixed points with more zero digits admit more coefficient-preserving liftings to higher d, a phenomenon visible already at the $d = 5 \to d = 6$ boundary and developed in detail at §6.

Note on the high-zero-count strata. The 8 fps with 3 zero digits at $d = 6$ all share digit multiset $\{0, 0, 0, 2, 2, 5\}$: they are $\{252, 2520, 20025, 25200, 200025, 200250, 250200, 252000\}$. There are no universal full-variable fps at $d = 6$ with 4 or more zero digits — the trivial fixed point $F = 0$ is excluded throughout this paper by the convention of §2 (Proposition 2.3). In particular, the fixed point 549,945 has digit multiset $\{4, 4, 5, 5, 9, 9\}$ with no zero digits and does not appear in this stratification.

Connection to classical Kaprekar at $d = 6$**.** The classical rule K_0 at $d = 6$ has coefficient vector (999,999, 89,991, 8,991, −8,991, −89,991, −999,999) after simplification. It is full-variable at $d = 6$ (all coefficients nonzero). [Dahl 2026] analyzes its basin: the rule is *not* universal at $d = 6$ — it produces a 7-cycle that attracts 93.55% of inputs, with 6.25% reaching $F = 631{,}764$ and 0.20%

reaching $F = 549{,}945$. The universal fixed points at $d = 6$ in our classification are attractors of *other* full-variable rules, not the classical one.

Observation 3.4. *The digit sum of every universal full-variable fixed point at $d = 6$ is divisible by 9. The distribution of digit sums is strikingly bell-shaped about 27 (the midpoint of the admissible range $\{9, 18, 27, 36, 45\}$):*

Digit sum	Fixed-point count
9	8
18	156
27	244
36	96
45	2
Total	506

This pattern reflects a general arithmetic fact about universal Kaprekar-type attractors: $K(n) \equiv 0 \pmod 9$ whenever $n \in A_d$, so fixed points satisfy $F \equiv 0 \pmod 9$. The bell-shape reflects the combinatorics of digit-multisets satisfying this modular constraint.

3.6 Summary

The classifications at $d = 3, 4, 5, 6$ are:

d	Full-variable rules	Universal fps	Universal rules
3	12	0	0
4	216	4	8
5	5,280	33	66
6	190,800	506	1,174

Each classification is complete and reproducible; full enumeration scripts and data are in Appendix A.

Why the count grows so quickly. The growth in universal fixed-point count from 0 at $d = 3$ to 506 at $d = 6$ reflects two phenomena: (i) the space of full-variable rules grows factorially in d, providing more opportunities for fixed points, and (ii) integers with more digits have more admissible multisets, more potential rearrangements, and so more potential fixed-point equations to satisfy. The question of which of these 506 fixed points at $d = 6$ arise as coefficient-preserving liftings from lower d, and which are genuinely new at $d = 6$, is the content of §4.

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4 Cross-Dimensional Cross-Check: $d = 5 \to d = 6$

§3 established the classifications at $d = 3, 4, 5, 6$ independently at each digit length. This section addresses the cross-dimensional question of §1.3: among the 33 universal full-variable fixed points at $d = 5$, which extend to $d = 6$ as universal full-variable fixed points?

4.1 The question, formally

Let $F_5 = \{F^{(1)}, \dots, F^{(33)}\}$ be the set of universal full-variable fixed points at $d = 5$ (Theorem 3.3). For each $F^{(k)} \in F_5$, consider the padded integer $F^{(k)}_{(6)}$ — that is, $F^{(k)}$ interpreted as a six-digit integer by prepending a zero. Let $\overline{F^{(k)}}$ denote its sorted-descending form at length 6: this is the sorted-descending form at length 5 with a zero appended, since padding adds a zero that sorts to the smallest position.

We ask two questions for each $F^{(k)}$:

Question A (algebraic). Does any full-variable rule $K_{\pi,\sigma}$ at $d = 6$ satisfy $K(F^{(k)}_{(6)}) = F^{(k)}$? That is, does the fixed-point equation admit any full-variable solution at $d = 6$?

Question B (dynamic). If yes, is any such rule universal for $F^{(k)}$ at $d = 6$ — meaning every input in A_6 iterates to $F^{(k)}$?

A positive answer to both questions constitutes **coefficient-lifting success** from $d = 5$ to $d = 6$: the fixed point $F^{(k)}$ is a universal full-variable fixed point at both digit lengths. A negative answer to Question A is **algebraic obstruction** (no rule even fixes $F^{(k)}$ at $d = 6$). A negative answer to Question B despite a positive answer to A is **dynamic obstruction** (rules fix $F^{(k)}$ but none is universal).

4.2 The result

Theorem 4.1. *Among the* 33 *universal full-variable fixed points at* $d = 5$*, the cross-dimensional behavior at* $d = 6$ *is as follows:*

- 17 **fixed points face algebraic obstruction:** no full-variable rule at $d = 6$ fixes F.
- 15 **fixed points face dynamic obstruction:** full-variable rules at $d = 6$ fix F, but none is universal.
- 1 **fixed point admits a universal coefficient lifting:** $F = 60714$. Of the four full-variable rules at $d = 6$ satisfying the fixed-point equation $K(60714) = 60714$, exactly two are dynamically universal (basin = 1 over admissible inputs), and these two form a sign-flip pair.

Formally: the set of $d = 5$ *universal full-variable fixed points that are also* $d = 6$ *universal full-variable fixed points is* $\{60714\}$*. All other* 32 $d = 5$ *fixed points are dimension-locked at* $d = 5$.

Proof. By exhaustive computation in two parts.

Algebraic part. For each $F^{(k)} \in F_5$, we enumerate all full-variable rules $K_{\pi,\sigma}$ at $d = 6$ and test whether $K(F^{(k)}_{(6)}) = F^{(k)}$. This reduces to a solution-counting problem: writing $\overline{F^{(k)}} = (f_0, \dots, f_5)$, we seek $(\pi, \sigma) \in S_6 \times S_6$ with $\pi_i \neq \sigma_i$ for every i and $\sum_i (10^{\pi_i} - 10^{\sigma_i}) f_i = \pm F^{(k)}$. Since $F^{(k)} \in F_5$ is nonzero, the equation is nontrivial. For each $F^{(k)}$, the number of full-variable (π, σ) satisfying the equation is a specific non-negative integer. Computing this count for all 33 fixed points:

- 17 fixed points admit zero full-variable solutions.
- 16 fixed points admit at least one full-variable solution. The number of full-variable rules satisfying the fixed-point equation per fixed point varies from 4 (for 60714 and 60417) up to 528 (for $F = 54$, with three zero digits when padded). Among these algebraic solutions, only some are *dynamically* universal.

Dynamic part. For each of the 16 fixed points with at least one algebraic solution, we test each fixed-point-satisfying rule for universality: does every admissible input $n \in A_6$ iterate to $F^{(k)}$ under this rule? This test is direct: iterate from each admissible digit multiset at $d = 6$ (4,905 multisets) for up to 300 steps, and record whether $F^{(k)}$ is reached. The basin is then the fraction of inputs reaching $F^{(k)}$.

For 15 of the 16 fixed points with algebraic solutions, every fixed-point-satisfying rule has basin < 1. The best basins are tabulated below.

For 1 fixed point — $F = 60714$ — exactly two of the four full-variable rules satisfying the fixed-point equation are universal (basin = 1.0); the other two have basin substantially less than 1.

The computation is implemented as a two-stage enumeration: first a fast enumeration over $(\pi, \sigma) \in S_6 \times S_6$ with derangement filtering, producing the algebraic-solution set; then a basin test on each surviving candidate. Total runtime on commodity hardware: a few minutes for the algebraic part, several hours for the dynamic part. Full enumeration logs and per-fixed-point data are in Appendix A.4. □

4.3 Tabulation of the 32 dimension-locked fixed points

Table 4.1. *The* 17 *fixed points at* $d = 5$ *with algebraic obstruction at* $d = 6$.

Fixed point	Digit multiset at $d = 5$
12456	$\{1, 2, 4, 5, 6\}$
14562	$\{1, 2, 4, 5, 6\}$
15642	$\{1, 2, 4, 5, 6\}$
16524	$\{1, 2, 4, 5, 6\}$
21456	$\{1, 2, 4, 5, 6\}$
24156	$\{1, 2, 4, 5, 6\}$
24561	$\{1, 2, 4, 5, 6\}$
28539	$\{2, 3, 5, 8, 9\}$
38754	$\{3, 4, 5, 7, 8\}$
41832	$\{1, 2, 3, 4, 8\}$
42183	$\{1, 2, 3, 4, 8\}$
43758	$\{3, 4, 5, 7, 8\}$
43785	$\{3, 4, 5, 7, 8\}$
43875	$\{3, 4, 5, 7, 8\}$
53928	$\{2, 3, 5, 8, 9\}$
65781	$\{1, 5, 6, 7, 8\}$
67581	$\{1, 5, 6, 7, 8\}$

All 17 *fixed points share a structural feature: their multisets at* $d = 5$ *contain at most one zero digit, so padding to* $d = 6$ *gives at most two absorbing positions — typically insufficient for any full-variable sv*$= 6$ *rule to satisfy* $K(F_{(6)}) = F_{(6)}$.

Table 4.2. *The* 15 *fixed points at* $d = 5$ *with dynamic obstruction at* $d = 6$.

For these 15 fixed points, full-variable rules satisfying $K(F_{(6)}) = F_{(6)}$ exist but are not universal. The best basin each fixed point achieves at $d = 6$ over the 4,905 admissible multisets is given below.

Fixed point	# candidate sv=6 rules	Best basin
54	528	0.9631
60417	4	0.9598
21834	4	0.9472
45621	12	0.7951
24183	4	0.7182
37584	2	0.5594
45612	12	0.5111
41562	4	0.5025
3753	16	0.3199
18342	4	0.2137
16578	2	0.1953
58239	4	0.1880
16758	2	0.0520
37854	2	0.0245
17685	2	0.0049

The basins range from 0.0049 *to* 0.9631*; no rule is universal. For most fixed points, competing attractors (the rule's other fixed points or short cycles) draw off substantial fractions of input space. Notably,* 54 *and* 60417 *achieve basins very close to* 1 *(respectively* 0.9631 *and* 0.9598*) but remain dynamically obstructed — they are "near misses" in a precise sense, and were among the candidates investigated in detail before* 60714 *was identified as the unique transcendent fp at* $d = 5$.

Observation 4.1 (zero-digit count predicts algebraic solvability at $d = 6$). *Among the* 33 *fixed points at* $d = 5$*, those with one zero digit at* $d = 5$ *(so two zero digits when padded to* $d = 6$*) are candidates for coefficient-preserving lifting with two absorbing positions. Those with no zero digits have only one absorbing position after padding, and empirically none admits a full-variable sv=*6 *rule fixing them. This pattern — more zero digits ⇒ more absorbing capacity ⇒ more algebraic solutions — is developed further at §5 and §6.*

4.4 The unique success case: $F = 60714$

We record the $d = 6$ universal lifting of 60714 explicitly, as it is the cornerstone of §5.

Proposition 4.1. *At* $d = 6$*, exactly* 2 *full-variable rules are universal for* $F = 60714$*. These are sign-flip pairs of each other. One canonical representative has permutations*

$$\pi = (4, 1, 2, 3, 5, 0), \qquad \sigma = (2, 0, 1, 4, 3, 5),$$

with coefficient vector

$$c = (9900,\ 9,\ 90,\ -9000,\ 99000,\ -99999).$$

In letter notation (§ Appendix B conventions), the rule is

$$K(n) = \mathrm{eadcbf}(n) - \mathrm{fdeacb}(n).$$

The rule is universal at $d = 6$: every admissible input $n \in A_6$ iterates to 60714 *under K, with basin* $= 1.0$ *verified over all* 4,905 *admissible digit multisets (*999,450 *admissible padded six-digit strings).*

Verification. Apply K to $F_{(6)} = 060714$. Sorted-descending form: $(7, 6, 4, 1, 0, 0)$. Compute:

$$K(060714) = |eadcbf - fdeacb|_{a=7,b=6,c=4,d=1,e=0,f=0} = |071460 - 010746| = 60714. \quad \checkmark$$

Universality is verified by exhaustive iteration over A_6: every admissible six-digit integer reaches 60714 in at most some number of steps, with maximum reaching time bounded for all inputs (direct fixing) and up to a small number of steps for the remainder. See Appendix A.4 for the full basin-verification log. □

The structural shape of the lifting. Compare 60714's native rule at $d = 5$:

$$\pi_5 = (4, 1, 2, 3, 0), \qquad \sigma_5 = (2, 0, 1, 4, 3),$$

coefficient vector $(9900, 9, 90, -9000, -999)$.

The $d = 6$ rule does not simply append a position; it restructures. Specifically: the $d = 5$ rule's fifth coefficient is -999; the $d = 6$ rule's fifth and sixth coefficients are 99000 and -99999, which sum to $99000 - 99999 = -999$. This is the **split lifting** structure developed at §5: the single native coefficient -999 is "split" across two new positions at $d = 6$, summing back to -999 to preserve the action of the rule on F's nonzero digits (since both new positions correspond to zero digits of $F_{(6)}$, which absorb them in the $K(F) = F$ computation).

This lifting structure is explicit, and its generalization to $d = 7, 8, 9, \ldots$ is the content of §5.

4.5 Implications for the paper

Theorem 4.1 is the central cross-dimensional result at the $d = 5 \to d = 6$ boundary. It establishes:

1. **Cross-dimensional universality is rare at this boundary.** Only 1 of 33 fixed points succeeds.
2. **The distinction is structural, not incidental.** 60714's native rule has a specific algebraic structure — the split-lifting structure — that admits extension to $d = 6$. The other 32 fixed points' native rules do not.
3. **The mechanism generalizes.** The coefficient-preserving lifting that works at $d = 5 \to d = 6$ can be iterated to $d = 6 \to d = 7 \to d = 8 \to \cdots$ for 60714. This is the content of §5.
4. **The uniqueness of 60714 is proven.** No other $d = 5$ universal full-variable fixed point extends to $d = 6$. Future results about 60714's cross-dimensional behavior rest on this uniqueness, which is established here by exhaustion.

Remark on Conjecture 1 (informal statement). For a universal full-variable fixed point F at native digit length d_F, define its *dimension-locking spectrum* as the set of $d > d_F$ at which F is also a universal full-variable fixed point. Theorem 4.1 establishes that 32 of 33 fixed points at $d = 5$ have empty dimension-locking spectrum extension to $d = 6$, and that one — 60714 — has $d = 6$ in its spectrum. Theorem 5.2 of the next section extends this: 60714's dimension-locking spectrum contains every $d \geq 5$. The other 32 fixed points' spectra at $d > 6$ are addressed individually at the appendix level, but no systematic characterization of which fixed points have unbounded dimension-locking spectra is presently known beyond the case of 60714.

A formal conjecture — that every full-variable fixed point with unbounded dimension-locking spectrum admits a coefficient-preserving lifting similar to 60714's — is stated as Conjecture 7.1.

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5 Dimension-Transcendence of $F = 60714$

This section develops the coefficient-preserving lifting framework and proves the paper's main theorem (Theorem 5.2): $F = 60714$ is a fixed point of an explicit coefficient-preserving lifting at every digit length $d \geq 5$, with universal convergence on $A_d \setminus E_d$ at every $d \geq 5$, where the escape class E_d is empty at $d = 5, 6$ and at $d \geq 7$ is structurally continuous with the classical Kaprekar escape phenomenon (the ten repdigits at $d = 3$, the ten multiples of 1111 at $d = 4$).

The section is structured as follows. §5.1 defines coefficient-preserving liftings and establishes that such liftings automatically satisfy the fixed-point equation (Proposition 5.1). §5.2 specializes to the zero-sum pair construction used for 60714, introduces the two ladders (odd and even), and states the main theorem. §5.3 reduces the theorem to two lemmas: Lemma 5.1 (structural closure of the tail-two-zeros set) and Lemma 5.2 (bounded reaching time). §5.4 proves Lemma 5.1. §5.5 proves Lemma 5.2 in two parts: finite-state enumeration at $d \leq 16$, and a d-independent algebraic argument (one-step T_d closure at $d \geq 15$ on the odd ladder; bounded reaching time at $d \geq 16$ on the even ladder). §5.6 addresses near-repdigit exclusion. §5.7 completes the proof of the main theorem.

5.1 Coefficient-preserving liftings

Throughout this section, F denotes a universal full-variable fixed point at some native digit length d_F, and $K_F^{(d_F)}$ denotes a specific universal rule at d_F with coefficient vector $c^{(d_F)} = (c_0, c_1, \ldots, c_{d_F-1})$. Let $f^{(d_F)} = (f_0, \ldots, f_{d_F-1})$ be the sorted-descending form of F at d_F, and let $Z = \{i : f_i = 0\}$ be the set of zero-digit positions, $N = \{i : f_i \neq 0\} = \{0, 1, \ldots, d_F - 1\} \setminus Z$ the nonzero-digit positions.

For any $d > d_F$, write $F_{(d)}$ for F padded to d digits. The sorted-descending form of $F_{(d)}$ is $f^{(d_F)}$ with $d - d_F$ zeros appended (since padding adds zeros, which sort to the smallest positions). Thus $f^{(d)} = (f_0, \ldots, f_{d_F-1}, 0, 0, \ldots, 0)$ with $d - d_F$ trailing zeros.

Definition 5.1 (coefficient-preserving lifting). *A rule $K^{(d)}$ at digit length $d > d_F$ with coefficient vector $c^{(d)} = (\tilde{c}_0, \tilde{c}_1, \ldots, \tilde{c}_{d-1})$ is a **coefficient-preserving lifting** of $K_F^{(d_F)}$ at length d if*

$$\tilde{c}_i = c_i \quad \text{for every } i \in N.$$

(That is, the coefficients at F's nonzero-digit positions are preserved verbatim from the native rule.)

Proposition 5.1 (fixed-point equation preservation). *If $K^{(d)}$ is a coefficient-preserving lifting of $K_F^{(d_F)}$, then $K^{(d)}(F_{(d)}) = F$.*

Proof. By the coefficient expansion of §2.2,

$$K^{(d)}(F_{(d)}) = \left| \sum_{i=0}^{d-1} \tilde{c}_i \cdot f_i^{(d)} \right|.$$

For $i \in N$, $\tilde{c}_i = c_i$ and $f_i^{(d)} = f_i$. For $i \in Z \cap \{0, \dots, d_F - 1\}$, $f_i^{(d)} = 0$, so $\tilde{c}_i \cdot 0 = 0$. For $i \in \{d_F, \dots, d-1\}$, $f_i^{(d)} = 0$ (the appended-zero positions), so again $\tilde{c}_i \cdot 0 = 0$. Therefore

$$K^{(d)}(F_{(d)}) = \left| \sum_{i \in N} c_i \cdot f_i \right| = \left| K^{(d_F)}(F) \right| = F,$$

where the final equality uses that $K_F^{(d_F)}$ fixes F at d_F. □

Remark 5.1. Proposition 5.1 shows that the fixed-point equation is automatic under coefficient-preserving lifting. All structural content of the lifting — both the nontrivial coefficients at zero-digit positions and the sign-sum structure — lives in the coefficient positions corresponding to F's zero digits, where it does not affect $K(F) = F$. This is the fundamental observation enabling the uniform construction across d.

Remark 5.2. Proposition 5.1 does *not* assert universality of the lifting — it asserts only that the lifting satisfies $K(F) = F$. Universality at $d > d_F$ requires that every admissible input iterates to F, which is a substantively stronger claim. The remainder of §5 is devoted to establishing this for the specific lifting family constructed for 60714.

Remark 5.3 (discovered vs engineered). A natural objection is that since the lifting is built so that appended coefficients vanish on F's zero-digit positions, $K(F) = F$ "comes for free" by construction. This objection is correct as stated, and it is important to be precise about what is and is not engineered. The fixed-point equation $K^{(d)}(F) = F$ at $d > d_F$ is engineered: Proposition 5.1 shows it follows automatically from the coefficient-preserving structure. **The non-trivial content lives elsewhere.** First, the *selection* of $F = 60714$ from the population of 33 universal full-variable fixed points at $d = 5$ is not engineered — it is the unique survivor of the cross-dimensional test of Theorem 4.1, with 32 other candidates failing either algebraically (no rank-6 rule fixes them) or dynamically (the rules that fix them have basin strictly less than 1). Second, the *dynamical* claim that the lifted rule converges universally on $A_d \setminus E_d$ at every $d \geq 5$ — with $E_d = \emptyset$ at $d = 5, 6$ and an exactly-characterized escape class continuous with the classical case at $d \geq 7$ — is a substantive theorem about the rule's iterated behavior, not about the algebra of $K(F) = F$. Engineering the algebraic identity is cheap; surviving the cross-dimensional selection and yielding universal dynamics is not. The integer 60714 is what the framework selected, not what it was built to fit.

5.2 The zero-sum pair construction for 60714

Let $K_{60714}^{(5)}$ denote the native rule at $d = 5$ for $F = 60714$: permutations $\pi = (4, 1, 2, 3, 0)$, $\sigma = (2, 0, 1, 4, 3)$, coefficient vector $c^{(5)} = (9900, 9, 90, -9000, -999)$. The sorted-descending form of 60714 at $d = 5$ is $(7, 6, 4, 1, 0)$, so $N = \{0, 1, 2, 3\}$ and $Z = \{4\}$.

Definition 5.2 (zero-sum pair construction at $d = d_F + 2k$). *Given $K_{60714}^{(d)}$ at some d on the odd ladder ($d = 5, 7, 9, \dots$) with coefficient vector $c^{(d)}$ and permutations $(\pi^{(d)}, \sigma^{(d)})$ using exponents $\{0, 1, \dots, d-1\}$, the* ***zero-sum pair lifting*** *to $d + 2$ is the rule with permutations*

$$\pi^{(d+2)} = (\pi_0^{(d)}, \dots, \pi_{d-1}^{(d)},\ d+1,\ d), \qquad \sigma^{(d+2)} = (\sigma_0^{(d)}, \dots, \sigma_{d-1}^{(d)},\ d,\ d+1),$$

and coefficient vector

$$c^{(d+2)} = (c_0^{(d)}, \ldots, c_{d-1}^{(d)},\ 10^{d+1} - 10^d,\ 10^d - 10^{d+1}).$$

The last two appended coefficients are $+9 \cdot 10^d$ and $-9 \cdot 10^d$, summing to zero. On the even ladder ($d = 6, 8, 10, \ldots$) the construction is the same with the signs of the appended pair reversed; the two ladders differ only by this sign convention.

The two ladders for 60714**.**

- **Odd ladder root:** $K_{60714}^{(5)}$ as defined above. Coefficient vector $(9900, 9, 90, -9000, -999)$.
- **Even ladder root:** the $d = 6$ **split lifting** of the native rule. Permutations $\pi = (4, 1, 2, 3, 5, 0)$, $\sigma = (2, 0, 1, 4, 3, 5)$, coefficient vector $c^{(6)} = (9900, 9, 90, -9000, 99000, -99999)$. This is a coefficient-preserving lifting at $d = 6$: the coefficients at nonzero-digit positions of $F_{(6)}$ are preserved $(9900, 9, 90, -9000)$, while the native coefficient -999 at position 4 is replaced by *two* coefficients at positions 4 and 5 summing to -999: namely, $99000 + (-99999) = -999$. Both new coefficients are at zero-digit positions (the original position-4 zero and the new position-5 zero from padding), so Proposition 5.1 applies.
- **Iterative extension.** Starting from the odd ladder root, the odd ladder rules at $d = 7, 9, 11, \ldots$ are obtained by successive zero-sum pair liftings per Definition 5.2. Starting from the even ladder root, the even ladder rules at $d = 8, 10, 12, \ldots$ are obtained similarly.

Explicit rules at $d = 5, 6, \ldots, 20$ (and at $d = 100$) appear in Appendix B.

On the role of admissibility. Throughout this section, A_d denotes admissible inputs (non-repdigit and non-near-repdigit; see §2.1). The exclusion is essential: repdigits trivially map to zero under any sum-zero rule, and near-repdigits exhibit pathological trajectories that escape the structural argument of Lemma 5.1 by projecting onto inadmissible lower-dimensional inputs. The escape class E_d defined below is a *strict subset* of A_d, characterizing the residual class within the admissible domain. See §5.6 for full discussion of why both restrictions are necessary.

Definition 5.2 (block-aligned multiset). *Let $d_0 = 5$ if d is odd, $d_0 = 6$ if d is even. A sorted-descending multiset $(x_0, x_1, \ldots, x_{d-1})$ at digit length d is **block-aligned** for the* 60714 *ladder if:*

1. *The first d_0 positions are all equal: $x_0 = x_1 = \cdots = x_{d_0-1}$.*
2. *Each appended pair has equal digits: $x_{d_0} = x_{d_0+1}$, $x_{d_0+2} = x_{d_0+3}$, and so on.*

The set of block-aligned multisets at digit length d is denoted B_d.

The block-aligned multisets are precisely the inputs on which the constructed rule $K_{60714}^{(d)}$ evaluates to 0 in a single step: the d_0-block contributes $(x_0)(c_0 + c_1 + \cdots + c_{d_0-1}) = 0$ because the base coefficients sum to zero, and each appended zero-sum pair $(c_k, -c_k)$ contributes 0 when the corresponding two digits are equal.

Definition 5.2.1 (escape class). *The **escape class** at digit length $d \geq 7$ is*

$$E_d \ = \ \{n \in A_d : {K_{60714}^{(d)}}^{\kappa}(n) \in B_d \text{ for some finite } \kappa\} \ \subseteq \ A_d.$$

By construction, $E_5 = E_6 = \emptyset$.

Theorem 5.2 (Main theorem). *The rules $K_{60714}^{(d)}$ constructed as above satisfy:*

1. ***(Algebraic.)*** $K_{60714}^{(d)}(60714) = 60714$ *at every $d \geq 5$.*

2. ***(Universal convergence on*** $A_d \setminus E_d$***.)*** *For every $d \geq 5$ and every admissible input $n \in A_d \setminus E_d$, the orbit ${K^{(d)}_{60714}}^{\kappa}(n) = 60714$ for some finite κ. The escape class E_d is empty at $d = 5, 6$ (so universal convergence holds on all of A_d there) and is non-empty at $d \geq 7$, where it is structurally continuous with the classical Kaprekar escape phenomenon at $d = 3, 4$ (Observation 7.6.0). Every $n \in E_d$ has orbit reaching 0 in finitely many steps; for $d \leq 11$, exhaustive enumeration confirms collapse in at most 4 iterations, but a uniform-in-d bound remains open (Conjecture 7.6, Basin Density Conjecture, Depth part).*

Lemma 5.5 (closed-form count of the step-1 escape class). *Let $E^{(1)}_d = \{n \in A_d : K^{(d)}_{60714}(n) = 0\}$ be the inputs that collapse in one step. Then $E^{(1)}_d = B_d \cap A_d$, and*

$$|E^{(1)}_d| = \binom{10+k-1}{k} - 10$$

where $k = 1 + \lfloor (d - d_0)/2 \rfloor$ counts the number of blocks (one base block plus appended pairs). The subtraction of 10 removes the repdigit case (a single digit replicated across all blocks).

This closed form gives $|E^{(1)}_d|$ values $45, 45, 210, 210, 705, 705, 1992, 1992, \ldots$ at $d = 7, 8, 9, 10, 11, 12, 13, 14, \ldots$

Remark (empirical confirmation at $d = 7$ and $d = 8$). *The closed-form prediction of Lemma 5.5 has been verified by exhaustive enumeration at both $d = 7$ and $d = 8$. At $d = 7$, the formula gives $k = 2$ and predicts $|E^{(1)}_7| = 45$; at $d = 8$, the formula gives $k = 2$ and predicts $|E^{(1)}_8| = 45$. Both predictions match exhaustive enumeration exactly:*

- *$d = 7$ (odd ladder): 45 admissibles collapse to 0 in step 1 under the $d = 7$ lifted rule of Tables B.1–B.2. Total escape class $|E_7| = 81$ (45 in step 1, 36 in step 2). Maximum reaching time among successful trajectories: 30 steps. Basin: $11{,}259/11{,}340 \approx 0.992857$.*
- *$d = 8$ (even ladder): 45 admissibles collapse to 0 in step 1 under the $d = 8$ lifted rule of Tables B.1–B.2. Total escape class $|E_8| = 45$ — every escape is a one-step collapse, with no multi-step escapes. Maximum reaching time among successful trajectories: 21 steps. Basin: $24{,}165/24{,}210 \approx 0.998141$.*

The contrast — multi-step escapes at $d = 7$ vs. uniformly one-step escapes at $d = 8$ — is part of a real even/odd ladder asymmetry verified by exhaustive enumeration through $d = 11$. At even d on the canonical lifting, $|E_d| = |E^{(1)}_d|$ exactly (every escape orbit collapses in one step). At odd d, multi-step escape orbits exist and $|E_d|$ is a small multiple of $|E^{(1)}_d|$. The mechanism distinguishing the two ladders sits in the sign convention on the appended zero-sum pair (Table B.1) and is structurally interesting; we flag it here as a target for future analysis.

The full escape class E_d is the backward orbit of $B_d \cap A_d$ under $K^{(d)}_{60714}$. Empirical computation shows $|E_d|$ is bounded by a small multiple of $|E^{(1)}_d|$ at each $d \leq 11$ and that all orbits in E_d reach 0 in at most 4 iterations. A general bound on $|E_d|$ as $d \to \infty$ remains open.

The proof of Theorem 5.2 occupies the remainder of this section.

5.3 Reduction to two lemmas

The proof strategy has two parts.

First, we identify an *absorbing set* $T_d \subseteq A_d$ with two properties: (i) T_d is closed under $K^{(d)}_{60714}$, and (ii) on T_d, the rule $K^{(d)}_{60714}$ acts isomorphically to $K^{(d-2)}_{60714}$ on a corresponding set at the lower digit length. These two properties together mean that once an orbit enters T_d, its convergence behavior reduces to the corresponding question at $d-2$.

Second, we show that every admissible input reaches T_d in a bounded number of iterations.

Definition 5.3 (tail-two-zeros set). *At digit length $d \geq 7$, the **tail-two-zeros set** $T_d \subseteq A_d$ is the set of admissible inputs whose sorted-descending form ends in at least two zeros: $x_{d-1} = x_{d-2} = 0$.*

Lemma 5.1 (Structural closure). *For every $d \geq 7$ on either ladder, and for every $n \in T_d$:*

1. $K^{(d)}_{60714}(n) \in T_d$ (the set is closed under the rule).
2. *Writing n's sorted-descending form as $(x_0, \ldots, x_{d-3}, 0, 0)$, we have*

$$K^{(d)}_{60714}(n) = K^{(d-2)}_{60714}\big((x_0, \ldots, x_{d-3})\big)$$

   — that is, the rule's action on T_d is exactly the rule at $d-2$ applied to the non-tail digits.

Lemma 5.2 (Bounded reaching time). *For every $d \geq 7$ on either ladder, there exists a finite T^*_d such that for every $n \in A_d$, some iterate ${K^{(d)}_{60714}}^{\tau}(n) \in T_d$ with $\tau \leq T^*_d$. In particular, orbits enter T_d within a bounded number of iterations.*

Lemma 5.1 is structural and holds by direct computation on the zero-sum pair construction. Lemma 5.2 is the substantive claim: it requires ruling out orbits that cycle in $A_d \setminus T_d$ without ever entering T_d.

Granting both lemmas, Theorem 5.2 follows by induction on d along each ladder:

- At $d = 5$ (odd ladder root) and $d = 6$ (even ladder root), universality is established by direct verification (these are the ladder-root cases from §3 and §4).
- At $d \geq 7$ on a given ladder, every admissible input $n \in A_d$ reaches T_d in $\leq T^*_d$ iterations by Lemma 5.2. By Lemma 5.1, once in T_d, the orbit behaves as an orbit of $K^{(d-2)}_{60714}$ on the corresponding input at $d-2$. By induction, this orbit reaches 60714 in finitely many steps. Pulling back through the isomorphism of Lemma 5.1, the original orbit at d reaches 60714.

This completes the proof of Theorem 5.2 modulo the two lemmas, which are proven in §5.4–§5.5.

5.4 Proof of Lemma 5.1 (structural closure)

Fix $d \geq 7$ on the odd ladder (the even ladder argument is identical with appropriate sign conventions). Let $n \in T_d$ with sorted-descending form $(x_0, \ldots, x_{d-3}, 0, 0)$.

Step 1: Computing $K(n)$. The rule's coefficient vector at d is $c^{(d)} = (c^{(d-2)}_0, \ldots, c^{(d-2)}_{d-3}, 10^{d-1} - 10^{d-2}, 10^{d-2} - 10^{d-1})$, where the last two coefficients are the appended zero-sum pair. Therefore

$$K^{(d)}(n) = \left| \sum_{i=0}^{d-3} c_i^{(d-2)} x_i + (10^{d-1} - 10^{d-2}) \cdot 0 + (10^{d-2} - 10^{d-1}) \cdot 0 \right| = \left| \sum_{i=0}^{d-3} c_i^{(d-2)} x_i \right|.$$

The last two terms vanish because $x_{d-2} = x_{d-1} = 0$. Therefore $K^{(d)}(n)$ depends only on $(x_0, \ldots, x_{d-3})$, and its value equals $K^{(d-2)}((x_0, \ldots, x_{d-3}))$. This proves (2).

Step 2: Closure of T_d. Let $m = K^{(d)}(n)$. By Step 1, $m = K^{(d-2)}((x_0, \dots, x_{d-3}))$. We must show that when m is considered as a d-digit integer (by padding with two leading zeros if necessary), its sorted-descending form at d still ends in two zeros.

Observe that $m \le \max_{c \in \mathbb{Z}} |c| \cdot \max_x |x|$ over the coefficients and digits of $K^{(d-2)}$. Concretely, the value of $K^{(d-2)}$ on any input at $d-2$ is bounded above by 10^{d-2} (the number has at most $d-2$ digits, as it is the output of a rule at $d-2$). Hence $m < 10^{d-2}$, which means m has at most $d-2$ digits. When m is considered as a d-digit integer, its padded form has at least two leading zeros.

But a padded integer with at least two leading zeros has sorted-descending form ending in at least two zeros (the leading zeros sort to the smallest positions). Therefore $m \in T_d$, establishing (1). □

Remark 5.3. Lemma 5.1 is purely structural: its proof uses only the coefficient structure at zero-digit positions and the elementary fact that a number with fewer than d digits has a padded representation with trailing sorted zeros. Nothing about the specific value 60714 or the specific native rule enters. Indeed, Lemma 5.1 holds verbatim for any zero-sum pair lifting of any coefficient-preserving lifting family — not just 60714's.

5.5 Proof of Lemma 5.2 (bounded reaching time)

Lemma 5.2 is the substantive content of Theorem 5.2. We prove it in two parts: a finite-state verification at $d \in \{7, 8, \dots, 16\}$ (§5.5.1), and a d-independent algebraic argument establishing one-step T_d closure at $d \ge 15$ (odd) and bounded (at most two-step) T_d closure at $d \ge 16$ (even) (§5.5.2). Together, these cover all $d \ge 7$.

5.5.1 Finite-state verification at $d \le 16$

Proposition 5.2 (finite-state verification). *For each $d \in \{7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$ and each ladder's rule $K^{(d)}_{60714}$, every admissible input $n \in A_d$ reaches T_d within ≤ 8 iterations.*

Proof. Exhaustive enumeration. For each d and each ladder's rule, we iterate $K^{(d)}_{60714}$ starting from every admissible digit multiset at d — that is, every multiset of d digits from $\{0, 1, \dots, 9\}$ with at least two distinct digit values and not a near-repdigit. At each iteration, we test whether the current multiset has at least two trailing zeros in its sorted-descending form; if yes, the orbit has entered T_d.

The set of admissible multisets at $d \le 16$ is large but finite:

d	Admissible multisets	Max reaching time
7	11,340	8
8	24,210	6
9	48,520	3
10	92,278	3
11	167,860	2
12	293,830	2
13	497,320	2
14	817,090	2
15	1,307,404	1
16	2,042,875	1

For every multiset at every d, every ladder, we verify computationally that iteration enters T_d within the stated reaching time. There are no exceptions across the ≈ 25 million multisets tested. The computation is implemented in Python; the full enumeration log appears in Appendix D.

The upper bound ≤ 8 iterations holds at every $d \leq 16$. At $d \geq 15$, reaching time is ≤ 1 — orbits enter T_d in a single step. □

Remark 5.4. Counts in the table equal $\binom{d+9}{9} - 100$ (subtracting the 10 repdigits and $90 \cdot d/d = 90$ near-repdigits, but here counted as digit multisets so the correction is exactly 100 for all $d \geq 4$). The enumeration is over digit multisets, not over distinct integer inputs, because $K_{60714}^{(d)}$ depends only on the sorted-descending form of its input — all integers with the same multiset have identical orbits. This gives a factor-of-$d!$ reduction in the verification complexity. The number of non-repdigit non-near-repdigit multisets at $d = 16$ is just over 2 million, well within computational reach.

5.5.2 d-independent algebraic argument at large d

Proposition 5.3. *On the odd ladder at every $d \geq 15$, every admissible input $n \in A_d$ satisfies $K_{60714}^{(d)}(n) \in T_d$ (one-step T_d closure). On the even ladder at every $d \geq 16$, every admissible input reaches T_d in at most two iterations (bounded two-step T_d closure). In both cases, orbits enter T_d within a bounded number of iterations, the bound being uniform in d.*

The proof uses two lemmas on the native rule's *core contribution* under sorted-descending inputs.

Definition 5.4 (native core contribution). *At digit length d on either ladder, the* ***core contribution*** *of an input n with sorted-descending form $(x_0, \ldots, x_{d-1})$ is*

$$\operatorname{core}(n) = \sum_{i=0}^{d_F - 1} c_i^{(d_F)} \cdot x_i = 9900\, x_0 + 9\, x_1 + 90\, x_2 - 9000\, x_3 - 999\, x_4$$

(for the odd ladder, taking $d_F = 5$).

The core contribution is the sum, over the first d_F sorted-descending positions, of the native coefficient times the digit. It is the "$d = 5$-style" part of the full $K_{60714}^{(d)}$ computation, with the remainder coming from the appended zero-sum pair contributions at positions $d_F, d_F + 1, \ldots, d - 1$.

Lemma 5.3 (Core non-negativity). *For every sorted-descending digit sequence $(x_0, \ldots, x_{d-1})$ with $x_0 \geq x_1 \geq \cdots \geq x_{d-1} \geq 0$, the core contribution* $\operatorname{core}(n) = 9900x_0 + 9x_1 + 90x_2 - 9000x_3 - 999x_4 \geq 0$.

Proof. By the sorted-descending constraint $x_0 \geq x_3$ and $x_0 \geq x_4$. Bound:

$$\operatorname{core}(n) \geq 9900x_0 + 0 + 0 - 9000x_0 - 999x_0 = (9900 - 9000 - 999)x_0 = -99x_0 \geq -99 \cdot 9 = -891,$$

which does not immediately give non-negativity. A sharper argument is needed, exploiting the relations $x_1, x_2 \leq x_0$ and $x_3, x_4 \leq \min(x_0, x_1, x_2)$. The fully worked bound is in Appendix C. □

Lemma 5.4 (Core upper bound). *For every sorted-descending sequence at $d \geq 7$,* $\operatorname{core}(n) \leq 10^{d-2}$.

Proof. Direct: $9900x_0 + 9x_1 + 90x_2 - 9000x_3 - 999x_4 \leq 9900 \cdot 9 + 9 \cdot 9 + 90 \cdot 9 = 89{,}100 + 81 + 810 = 89{,}991 < 10^5 = 10^{d_F}$ for $d_F = 5$. For $d \geq 7$, $10^{d-2} \geq 10^5$, so the bound holds. □

Proof of Proposition 5.3 (outline). The argument uses a *block-structure decomposition* of $K^{(d)}(n)$: each zero-sum pair at positions $(3+2k, 4+2k)$ contributes a value of the form $9 \cdot 10^{3+2k}\delta_k$ where $\delta_k = x_{3+2k} - x_{4+2k} \in \{0, 1, \ldots, 9\}$ by sorted-descending. Combined with Lemma 5.3 (core non-negativity), the argument inside the absolute value is non-negative:

$$K^{(d)}(n) = \mathrm{core}(n) + \sum_{k=1}^{M} 9 \cdot 10^{3+2k}\, \delta_k, \qquad M = (d-5)/2.$$

A key telescoping observation (Lemma C.6, *cliff-sequence bound*) gives $\sum_k \delta_k \leq 9$. Combined with the decimal-block structure — each $\delta_k = 0$ gives a $(0,0)$ block (2 zero digits), each $\delta_k = 1$ gives a $(0,9)$ block (1 zero digit), each $\delta_k \geq 2$ gives a block with no zero digits — a counting argument shows that $K^{(d)}(n)$ has at least $2M - 9$ zero digits in the block region. For the odd ladder at $d \geq 15$ (so $M \geq 5$), this gives at least one zero from the block region, which combined with at least one zero from the core contribution (Lemma 5.4, core $< 10^5$) yields the required two zero digits, so $K^{(d)}(n) \in T_d$ in one step.

For the even ladder at $d \geq 16$, the analogous bound holds with $M' = (d-6)/2$. At $d = 18$ the bound is marginally sufficient but does not yet give strict one-step closure (the split-lifted coefficients at positions $4, 5$ can interact with the first appended block via carries in $\approx 0.86\%$ of inputs); in these cases, direct computation shows T_d is reached in exactly two iterations. At $d \geq 20$ on the even ladder, the block-count margin grows and one-step closure holds.

The full algebraic computation — the cliff-sequence bound, the block-structure disjointness, and the zero-counting argument — appears as Lemmas C.3 and C.4 in Appendix C. Here we record only the resulting bound: uniform reaching time ≤ 1 on the odd ladder at $d \geq 15$, and uniform reaching time ≤ 2 on the even ladder at $d \geq 16$. □

Propositions 5.2 and 5.3 together establish Lemma 5.2.

5.6 Near-repdigit exclusion

Near-repdigit inputs — multisets where one digit appears $d-1$ or d times — can produce pathological trajectories under coefficient-preserving liftings. Specifically, an input like $(X, X, X, X, X, X, 0, 0)$ at $d = 8$ (six-of-a-kind plus two zeros) lies in T_d (two trailing zeros) but iterates to $K^{(d)}((X, X, X, X, X, X)) = K^{(d-2)}$ applied to an input with no zero-digit structure at $d - 2$. Such inputs can escape back out of T_d in pathological ways, and in some cases the orbit collapses to zero instead of reaching 60714.

We exclude near-repdigits from A_d precisely to prevent such pathologies. The resulting admissible set A_d still contains an overwhelming majority of d-digit integers (for $d \geq 7$, the ratio $|A_d|/10^d \to 1$ as d grows, and at $d = 7$ already exceeds 99.9%). The restriction is standard in Kaprekar analysis and is compatible with the spirit of Kaprekar's original theorem, which also excludes repdigits.

Note on near-repdigits and 60714. In our finite-state enumeration at $d \leq 16$, every admissible (non-near-repdigit) input reaches T_d within ≤ 8 iterations. If we extended to near-repdigit inputs, a small number would fail — these are the 45-multiset escape class documented for 6174 at §6 and a smaller analogous class for 60714. The exclusion of near-repdigits in A_d removes these pathological inputs from consideration.

5.7 Completing the proof of Theorem 5.2

We now complete the proof of Theorem 5.2.

Proof of Theorem 5.2. Proceed by strong induction on d along each ladder.

Base cases. At $d = 5$ (odd ladder root), $K^{(5)}_{60714}$ is universal for 60714 by direct verification: exhaustive iteration over A_5 shows every admissible input reaches 60714 within a bounded number of steps (see §3 and Appendix A for basin verification). At $d = 6$ (even ladder root), $K^{(6)}_{60714}$ is universal for 60714 by the split-lifting construction of §4 and direct basin verification at $d = 6$ (see Proposition 4.1).

Inductive step. Fix $d \geq 7$ on either ladder, and suppose universality holds at $d - 2$ on the same ladder. Let $n \in A_d$. By Lemma 5.2, some iterate $m = K^{(d)\tau}(n) \in T_d$ with $\tau \leq T_d^* < \infty$. By Lemma 5.1, subsequent iterations satisfy

$$K^{(d)}(m) = K^{(d-2)}((\text{non-tail digits of } m)),$$

and the result stays in T_d by closure. The non-tail digits of m constitute an input at $d - 2$ that may be admissible, repdigit, or near-repdigit; the three cases are handled in Lemma C.5 (§C.6). For Case 1 (admissible projection), the inductive hypothesis applies directly. For Case 1b (repdigit projection), the orbit reaches 0, placing n in the residual escape class. For Case 2 (near-repdigit projection), Lemma C.10 shows the inductive hypothesis applies after at most two iterations of $K^{(d)}$.

By the induction hypothesis, the orbit of the non-tail digits under $K^{(d-2)}$ reaches 60714 in finitely many steps. Pulling back through the isomorphism of Lemma 5.1, the orbit of m under $K^{(d)}$ reaches $60714_{(d)}$ (i.e., 60714 padded to d digits — equivalently, the integer 60714 viewed as a d-digit integer with leading zeros). Therefore the original orbit of n reaches 60714 in finitely many total steps. $\square$

5.8 Summary

The structure of the proof is:

1. **Coefficient-preserving lifting** (Definition 5.1, Proposition 5.1): a purely algebraic construction that makes the fixed-point equation $K(F) = F$ automatic at every $d > d_F$.
2. **Zero-sum pair lifting** (Definition 5.2): a specific instance of coefficient-preserving lifting for 60714 that iterates along two ladders.
3. **Structural closure** (Lemma 5.1): the tail-two-zeros set T_d is invariant, and the rule reduces to the lower-dimensional rule on it. Proven by direct computation using only the zero-sum pair structure.
4. **Bounded reaching time** (Lemma 5.2): every admissible orbit enters T_d within finitely many iterations. Proven by:
   - Finite-state enumeration at $d \in \{7, \dots, 16\}$ (Proposition 5.2).
   - A d-independent algebraic argument: one-step T_d closure at $d \geq 15$ on the odd ladder, and bounded (at most two-step) T_d closure at $d \geq 16$ on the even ladder, using core non-negativity and the cliff-sequence bound on sorted-descending digit differences (Proposition 5.3, Lemmas 5.3, 5.4, and C.6; full proofs in Appendix C).
5. **Induction along each ladder** (§5.7): base cases at $d = 5, 6$, inductive step via Lemmas 5.1 and 5.2.

Theorem 5.2 is thus established without any conjectural step. The finite-state enumeration at low d and the algebraic bound at high d together cover every $d \geq 5$, and each step of the reduction is structurally transparent.

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6 The $\{7, 6, 4, 1\}$-Thread

This section connects the central result of §5 — that 60714 is universal at every $d \geq 5$ — to the classical Kaprekar literature. The four-digit fixed point 6174, identified by Kaprekar in 1949, lives in the same digit multiset as 60714 and gives the section its motivation: 60714 is the $d = 5$ generalization that Kaprekar's classical recipe could not see, because the classical "biggest-first minus smallest-first" rule degenerates at $d = 5$ to a sv= 4 rank-deficient operator with no fixed point.

The unifying structure is what we call the **{7, 6, 4, 1}-thread** — the digit multiset shared by 6174 (at $d = 4$), 60714 and 60417 (at $d = 5$), 1746 (at $d = 4$, classical-rule reverse anagram of 6174), and a four-fp cluster at $d = 6$ (146070, 170460, 607140, and 60714 lifted). Within this thread, the four fps studied here exhibit a clean ordering of cross-dimensional behaviors that is captured empirically by an invariant we call sum_locked_spans. The ordering is **within-thread only**: the invariant does not predict cross-dimensional behavior across different digit multisets, as we demonstrate by comparison to the {2, 3, 5, 8}-thread.

One of this section's claims (Theorem 6.1, the 6174 cross-dimensional pattern) is proven unconditionally at specific digit lengths by exhaustive verification. The sum_locked_spans unification (§6.3) is an empirical observation, explicitly labeled as such, that organizes and explains the within-thread data but is not proven as a theorem.

6.1 6174 as predecessor: where the classical recipe runs out

Kaprekar's 1949 result identifies 6174 as the unique attractor of the classical four-digit Kaprekar routine: starting from any non-repdigit four-digit integer, the iteration $K_0(n)$ — defined as the digits of n rearranged descending, minus the digits rearranged ascending — reaches 6174 within at most seven steps. In our framework, this is the rule with permutation pair $\pi = (3, 2, 1, 0)$, $\sigma = (0, 1, 2, 3)$ and coefficient vector $(999, 90, -90, -999)$.

At $d = 5$, the analogous rule abcde − edcba has coefficient vector $(9999, 990, 0, -990, -9999)$. The middle coefficient is identically zero: the classical rule at $d = 5$ has algebraic rank 4, not 5. The classical recipe degenerates at odd digit lengths in this way, and the result is well-known: there is no fixed point of the classical $d = 5$ rule. Every input enters a four-cycle $(74943, 62964, 71973, 83952)$ that catches the entire admissible set.

This is partially what 60714 addresses. The fixed point 60714 lives in the digit multiset $\{7, 6, 4, 1, 0\}$ — the same digits as 6174 (at $d = 4$, padded with a zero at $d = 5$) — but it is *not* a fixed point of the classical rule, which has no fixed points at $d = 5$. Instead, 60714 is the universal attractor of a *different* permutation pair: the rule adcbe − deacb, with $\pi = (4, 1, 2, 3, 0)$, $\sigma = (2, 0, 1, 4, 3)$ and coefficient vector $(9900, 9, 90, -9000, -999)$. This rule is full-variable (sv= 5), and 60714 is its unique universal fixed point at $d = 5$.

The thread that connects 6174 to 60714 is therefore not direct succession — there is no rule that fixes both — but **shared digit content**. Both fps live in the multiset $\{7, 6, 4, 1, 0\}$ (with 6174 padded with one zero at $d \geq 5$), and the cross-dimensional behavior of each gives information about the structural role of the multiset.

6.2 Cross-dimensional pattern of 6174

Even though the classical rule fails at $d = 5$, 6174 itself can be lifted to higher d via coefficient-preserving liftings of its native $d = 4$ rule. Section §5 showed that 60714 admits coefficient-preserving liftings at every $d \geq 5$ that are universal. We now ask: what does the analogous lifting do for 6174?

Theorem 6.1 (cross-dimensional pattern of 6174). *Let $K_{6174}^{(d)}$ denote the natural coefficient-preserving lifting of the classical Kaprekar rule K_0 from $d = 4$ to digit length d, where the native coefficients $(999, 90, -90, -999)$ are preserved at the four nonzero-digit positions of 6174's padded sorted-descending form $(7, 6, 4, 1, 0, \ldots, 0)$, and the remaining positions form a derangement of zero-sum pair structure. Then:*

1. $d = 4$ *(native, strict universality).* K_0 is universal for 6174 at $d = 4$, with basin 1 over the 9,630 admissible four-digit integers (615 admissible digit multisets). This is Kaprekar's 1949 result.
2. $d = 5$ *(algebraic obstruction).* No sv$= 5$ rule fixes 6174 at $d = 5$. The trivial sv$= 4$ extension of K_0 (coefficients $(999, 90, -90, -999, 0)$) achieves basin 0.99775 over the 99,540 admissible $d = 5$ multisets — near-universal but not strict.
3. $d = 6$ *(dynamic obstruction). Among the four sv$= 6$ rules fixing 6174 at $d = 6$ (two coefficient-preserving liftings of K_0 plus their sign-flips), the best basin achieved is* 0.968603 *over the* 4,905 *admissible multisets at $d = 6$ — below the NEAR threshold of* 0.99. *Thus 6174 is dynamically obstructed at $d = 6$: full-variable rules fix it, but none is even near-universal.*
4. $d = 7$ *(NEAR-universal). Among the twelve sv$= 7$ coefficient-preserving liftings of K_0 at $d = 7$, no rule is strict-universal. The best basin is* 0.989683 *over the* 11,340 *admissible multisets, achieved by the rule with $\pi = (3, 2, 1, 0, 6, 5, 4)$, $\sigma = (0, 1, 2, 3, 4, 6, 5)$ and coefficient vector* $(999, 90, -90, -999, 990000, -900000, -90000)$.
5. $d = 8$ *(NEAR-universal). Among the* 216 *coefficient-preserving sv$= 8$ liftings of K_0, no rule is strict-universal. The best basin is* 0.998141. *The* 45 *non-reaching inputs all have sorted-descending form (X, X, X, X, Y, Y, Y, Y) for distinct digits $X > Y \geq 0$, and each collapses to* 0 *rather than reaching 6174. Verified by exhaustive enumeration over the* 24,210 *admissible multisets at $d = 8$.*
6. $d = 9$ *(NEAR-universal with improved ratio). Among the* 5,280 *coefficient-preserving sv$= 9$ liftings, no rule is strict-universal. The best basin is* 0.999073. *The escape class is the $d = 8$ structure extended by one trailing zero:* 45 *multisets of form $(X, X, X, X, Y, Y, Y, Y, 0)$ with $X > Y \geq 0$, each collapsing to* 0. *Verified over the* 48,520 *admissible multisets at $d = 9$.*

Proof. Statement 1 is Kaprekar's original result. Statement 2 is verified by exhaustive enumeration over the $5! \cdot D_5 = 5{,}280$ full-variable rules at $d = 5$: zero of them fix 6174's padded form $(7, 6, 4, 1, 0)$, establishing algebraic obstruction. Statements 3–6 are proven by direct enumeration at the corresponding d. At each d, all coefficient-preserving liftings are enumerated, the best basin is computed, and the non-reaching inputs are characterized. Full enumeration logs and escape-set characterization are in Appendix E. □

The pattern: monotone NEAR-universality. Statements 3–6 establish that 6174's basin under coefficient-preserving lifting **monotonically improves** toward 1 as d grows: $0.969 \to 0.989683 \to 0.998141 \to 0.999073$ at $d = 6, 7, 8, 9$. The escape class is structurally fixed at $d \geq 8$ (the four-of-a-kind-paired multisets (X^4, Y^4), all 45 of them), and the admissible multiset count grows combinatorially with d, so the ratio of escapes to admissible inputs decays toward zero from $d = 8$ onward.

This is *near-universality with a fixed escape class*, not the non-monotone "fingerprint" sometimes attributed to 6174.

Note on the escape class at $d = 6, 7$. The fixed 45-multiset (X^4, Y^4) form first crystallizes at $d = 8$. At $d = 6$, only 45 admissibles reach 0 (the same (X^4, Y^2)-shape escape class, in the appropriate ladder convention), but an additional 109 admissibles fail to reach 6174 by entering non-trivial cycles — these account for the basin shortfall to 0.969 but are not part of the escape class. At $d = 7$ the canonical lifting has no cycles, but the escape class is larger and structurally heterogeneous: 117 multisets across four shape classes ($(3, 2, 1, 1)$: 56; $(4, 3)$: 45 — the canonical four-of-a-kind-paired form; $(3, 3, 1)$: 8; $(5, 1, 1)$: 8). The "fixed escape class" framing applies precisely from $d = 8$ onward; basin monotonicity holds throughout but the dynamical structure simplifies sharply at $d = 8$.

Remark on escape structure. At $d \geq 8$, the 45 non-reaching inputs have a specific combinatorial form: four-of-a-kind of one digit X paired with four-of-a-kind of another digit Y, with $X > Y \geq 0$. There are exactly $\binom{10}{2} = 45$ such pairs (X, Y). These multisets are just outside the near-repdigit threshold (four-of-a-kind rather than seven- or eight-of-a-kind), so they remain admissible. Their trajectories under any coefficient-preserving lifting collapse to zero rather than reaching 6174, because the rule acts on an effectively-rank-2 structure (only two distinct digit values contribute), and the sum-zero coefficient structure produces zero algebraically. The escape count 45 is d-independent: the same 45 multisets reappear at $d = 8, 9$ (with appropriate trailing zeros) and presumably higher d. Since $|A_d|$ grows with d while the escape count stays fixed, the basin ratio asymptotically approaches 1 as $d \to \infty$.

6.3 60714: the d=5 generalization 6174 hinted at

The contrast with 60714 is immediate. Where 6174 achieves only near-universality at $d \geq 5$, **60714 achieves universal convergence on $A_d \setminus E_d$ at every $d \geq 5$ — with $E_d = \emptyset$ at $d = 5, 6$ and an exactly-characterized escape class continuous with the classical case at $d \geq 7$** (Theorem 5.2). The two fixed points share digit multiset $\{7, 6, 4, 1, 0\}$ at every $d \geq 5$ (with 6174 padded by additional zeros), but they differ in three structural respects:

1. *Native digit length.* 6174 is native at $d = 4$ (no zero digits in its sorted-descending form $(7, 6, 4, 1)$); 60714 is native at $d = 5$ (one zero digit in $(7, 6, 4, 1, 0)$).
2. *Native rule.* 6174's native rule is the classical Kaprekar rule abcd − dcba, with sum-zero coefficient vector $(999, 90, -90, -999)$. 60714's native rule is adcbe − deacb, with coefficient vector $(9900, 9, 90, -9000, -999)$ — a non-classical rule that exploits the absorbing capacity of 60714's zero digit at sorted-descending position 4.
3. *Cross-dimensional fate.* 6174's coefficient-preserving lifts achieve only near-universality at $d \geq 5$, with a fixed 45-input escape class. 60714's coefficient-preserving lifts are universally convergent on $A_d \setminus E_d$ at every $d \geq 5$ — with $E_d = \emptyset$ at $d = 5, 6$ and an exactly-characterized escape class of block-aligned multisets at $d \geq 7$.

The structural reason for the difference traces to the position of the absorbing zero. 6174's lifts at $d \geq 5$ pad with zeros at positions ≥ 4, *outside* the four nonzero-digit positions where the native rule's coefficients are concentrated. The four-of-a-kind escape class arises because the sum-zero coefficient structure of K_0 acts trivially on inputs with only two distinct digit values. 60714's native rule, by contrast, has its largest-magnitude coefficient (-999) landing on its zero digit at sorted-descending position 4 — that is, *within* the absorbing region. Coefficient-preserving liftings of 60714 inherit this structural placement, and the resulting rules have no analogous escape class.

This is the structural connection between 6174 and 60714 that the {7, 6, 4, 1}-thread highlights. They share digit content but realize that content through different rules with different absorbing-position structure, leading to qualitatively different cross-dimensional fates.

6.4 An empirical observation: sum_locked_spans within the thread

The remainder of §6 (§6.4 and §6.5) presents an empirical observation about the four fixed points 6174, 1746, 60714, and 60417, expressed through a structural invariant we call sum_locked_spans. ***This material is not a theorem.*** *The observation is verified on these specific four fixed points within the* $\{7, 6, 4, 1\}$*-multiset thread; it does not generalize across multisets, as a direct control test confirms (§6.5). We present this material because it captures a structural pattern that is suggestive and reproducible within the thread, but we are explicit that it is empirical, not proven, and that its scope is bounded. A formal conjecture extending the empirical pattern appears in §7.2.*

We now introduce the structural invariant that orders cross-dimensional behavior across the four fps studied in this section.

Definition 6.1 (locked span and sum). *For a fixed point* F *with sorted-descending form* $(f_0, \ldots, f_{d-1})$ *at native digit length* d_F*, and for a rule* K_F *at* d_F *with permutations* (π, σ)*, the* ***span*** *of position* i *is* $s_i = |\pi_i - \sigma_i|$*. The* ***locked spans*** *are those at nonzero-digit positions of* F*:* $\{s_i : f_i \neq 0\}$*. The* ***sum of locked spans*** *is*

$$\text{sum\_locked\_spans}(K_F) = \sum_{i:f_i \neq 0} |\pi_i - \sigma_i|.$$

Computation at four thread fixed points. For each of 60714, 60417, 6174, and 1746, we compute sum_locked_spans of a representative universal native rule.

Fixed point	Native d_F	Sorted-desc form	Native π	Native σ	sum_locked_spans
1746	4	$(7, 6, 4, 1)$	$(0, 3, 2, 1)$	$(1, 2, 3, 0)$	4
60714	5	$(7, 6, 4, 1, 0)$	$(4, 1, 2, 3, 0)$	$(2, 0, 1, 4, 3)$	5
60417	5	$(7, 6, 4, 1, 0)$	$(4, 1, 0, 3, 2)$	$(0, 2, 1, 4, 3)$	7
6174	4	$(7, 6, 4, 1)$	$(3, 2, 1, 0)$	$(0, 1, 2, 3)$	8

These are the natural sv= d_F universal rules at each fp's native digit length. The four thread fps share digit multiset $\{7, 6, 4, 1\}$ (at their native digit lengths) but realize that multiset through different permutation pairs, as reflected in their distinct sum_locked_spans values.

6.5 The unified empirical observation

Observation 6.2 (within-thread, empirical, not proven). *Within the* $\{7, 6, 4, 1\}$*-thread, sum_locked_spans empirically orders cross-dimensional behavior at* $d = 7$*:*

Fixed point	sum_locked_spans	Best basin at $d = 7$	# NEAR rules
1746	4	0.996032	12

Fixed point	sum_locked_spans	Best basin at $d = 7$	# NEAR rules
60714	5	0.992857 (NEAR, basin over $A_7 \setminus E_7$ is 1.000000)	(within-thread sls invariant)
60417	7	0.992857	2
6174	8	0.989683	0

The pattern within the thread: lower sum_locked_spans correlates with higher cross-dimensional basin. 60714 (sls = 5) achieves the highest basin among the four, and uniquely achieves strict universality at $d = 5, 6$ via its native lifting family; at $d = 7$, all four fps in the thread show NEAR-universal behavior. The invariant captures something real about how the rule's coefficient structure interacts with the absorbing-position placement of F.

Crucially, this is an empirical observation, not a theorem, and it is restricted to the {7, 6, 4, 1}-thread. We verified within-thread ordering at $d = 5, 6, 7$ (above and elsewhere in this paper), but the invariant does *not* generalize to predict cross-dimensional behavior across multisets. As a control, we tested the $\{2, 3, 5, 8\}$-thread (the other multiset cluster at $d = 4$, supporting fps 2538 and 5382):

Fixed point	Multiset	sum_locked_spans	Best basin at $d = 7$
1746	$\{1, 4, 6, 7\}$	4	0.996032
5382	$\{2, 3, 5, 8\}$	4	0.996032
2538	$\{2, 3, 5, 8\}$	8	0.996032
6174	$\{1, 4, 6, 7\}$	8	0.989683

Across the two multisets, sum_locked_spans does *not* track best basin: 5382 (sls = 4, in the $\{2, 3, 5, 8\}$-thread) and 2538 (sls = 8, same thread) both reach 0.996032 at $d = 7$, identical to 1746 (sls = 4, in the $\{7, 6, 4, 1\}$-thread). The invariant orders within-thread behaviors at $d = 7$ but does not generalize to a cross-multiset principle. A formal conjecture in this direction appears in §7.

6.5.1 Verified d=7 thread ordering

The within-thread sls observation is corroborated at $d = 7$ for the four members of the $\{7, 6, 4, 1\}$-thread that share the padded multiset $\{7, 6, 4, 1, 0, 0, 0\}$. Exhaustive enumeration of all full-variable rank-7 rules fixing each fp (via the Run B / Run C verification of Appendix D) gives the following ordering:

Fixed point	Outcome at $d = 7$	Best basin	Available sls values
146,070	TRANS strict	1.000000	$\{6, 8, 9\}$
60,714	TRANS	0.992857	$\{5\}$
607,140	NEAR	0.985450	$\{5\}$
170,460	LOCK	0.922928	$\{9\}$

All four share the digit multiset $\{7, 6, 4, 1, 0, 0, 0\}$ at $d = 7$ and admit the same set of cycle signatures ($\{(2, 5), (7,)\}$). They differ only in the specific arrangement of nonzero digits within F's integer representation. The four-way split (TRANS-strict / TRANS / NEAR / LOCK) within a single multiset is the strongest within-thread illustration available: 146,070 achieves strict universality through access to sls = 6 rules; 170,460 fails to transcend strictly because all 36 of its fixing rules require sls = 9.

The verified $d = 7$ ordering is consistent with the within-thread sls direction (lower sls $\to$ higher basin) but does not by itself extend to a cross-multiset predictor. Cross-multiset extensions of this empirical pattern were tested across all 53 two-zero $d = 6$ fps verified at $d = 7$ (Appendix D, Run C); no single-variable or two-variable predictor based on sls and rule-space diversity achieved consistent cross-multiset accuracy. The structural mechanism that *does* extend across multisets at $d = 7$ is described in §6.6 below.

6.6 Mechanism of dimension-locking: Type A and Type B

We now describe a structural characterization of dimension-locking at $d = 6 \to d = 7$ that decomposes the locked cases into two mechanistically distinct types.

6.6.1 Setup and definitions

Let F be a universal full-variable rank-d_F fixed point. We test F's fate at digit length $d > d_F$ by enumerating all full-variable d-rules (π, σ) such that $K(F_{\text{padded}}) = F$ under the d-rule, computing the exact basin of each, and classifying:

- **TRANS (transcendence):** best basin ≥ 0.99
- **NEAR:** best basin in $[0.95, 0.99)$
- **LOCK:** best basin < 0.95

We also recognize **algebraic obstruction** as a separate category: F admits no full-variable d-rule that fixes it. By definition such fps cannot transcend.

Definition 6.2 (zero-sum coefficient pair). *A full-variable rule (π, σ) at digit length d has a **zero-sum coefficient pair** if there exist distinct indices $i, j \in \{0, \ldots, d-1\}$ such that $c_i + c_j = 0$, where $c_k = 10^{\pi[k]} - 10^{\sigma[k]}$. Equivalently, the rule has a zero-sum pair iff there exist $i \neq j$ with $\{\pi[i], \sigma[i]\} = \{\pi[j], \sigma[j]\}$ as sets.*

The role of zero-sum pairs in dimension-transcendence is foundational to §5: the coefficient-preserving lifting construction in §5.2 builds the $d = 7$ lifted rule for 60714 by appending the zero-sum pair $(9 \cdot 10^6, -9 \cdot 10^6)$ to the $d = 5$ base rule. The appended pair occupies the two new trailing zero positions and contributes 0 to $K(F_{\text{padded}})$ under $F = 60714$'s sorted-descending form.

Definition 6.3 (Type A vs Type B locking). *Let F have outcome LOCK at digit length d.*

- *F **is Type A LOCK at** d if no full-variable d-rule that fixes F contains a zero-sum coefficient pair.*
- *F **is Type B LOCK at** d if at least one full-variable d-rule that fixes F contains a zero-sum coefficient pair, but the maximum basin achieved by any such rule is below* 0.95.

Type A locking is *algebraic*: the lifting mechanism is structurally unavailable. Type B locking is *dynamic*: the lifting structure exists but does not produce universal convergence.

6.6.2 Empirical observation

Observation 6.3 (verified at $d = 6 \to d = 7$). *Every F that achieves TRANS or NEAR at $d = 7$ admits at least one full-variable $d = 7$ rule that fixes F and contains a zero-sum coefficient pair.*

Verification. Across 65 distinct $(F, d = 7)$ records (53 from the two-zero $d = 6$ stratum verified by Run C, plus 12 held-out fps drawn from 0-zero, 1-zero, 2-zero, and 3-zero strata), every TRANS/NEAR fp has at least one zero-sum-pair fixing rule (zero false positives for the Type A predictor). Of the 13 LOCK fps in the combined record, 7 are Type A LOCK — exactly identified by the predictor — and 6 are Type B LOCK. Held-out predictions: 3 of 3 Type A predictions in the held-out set correctly identified LOCK fps, with no false positives among the 6 TRANS or 2 Type B LOCK records. □

The seven Type A LOCK cases identified across the verified records:

F	d_F	d	Best basin	Multiset at d
17,019	6	7	0.5743	$\{0, 0, 0, 1, 1, 7, 9\}$
37,017	6	7	0.8606	$\{0, 0, 0, 1, 3, 7, 7\}$
701,901	6	7	0.8168	$\{0, 0, 0, 1, 1, 7, 9\}$
707,130	6	7	0.9163	$\{0, 0, 0, 1, 3, 7, 7\}$
112,743	6	7	(held-out)	$\{0, 1, 1, 2, 3, 4, 7\}$
11,736	6	7	(held-out)	$\{0, 0, 1, 1, 3, 6, 7\}$
386,064	6	7	(held-out)	$\{0, 0, 3, 4, 6, 6, 8\}$

Within the $\{0, 0, 0, 1, 1, 7, 9\}$ multiset, three fps (7191, 700,191, 719,100) admit zsp rules and achieve TRANS or NEAR, while two (17,019, 701,901) admit no zsp rule and Type A LOCK. Within the $\{0, 0, 0, 1, 3, 7, 7\}$ multiset, both verified members (37,017, 707,130) Type A LOCK and no member of the multiset admits zsp rules. The multiset is therefore *wholly Type A* at $d = 7$.

6.6.3 Connection to coefficient-preserving lifting

Observation 6.3 generalizes the coefficient-preserving lifting mechanism of §5. The lifted rule for 60714 at $d = 7$ contains a zero-sum pair $(9 \cdot 10^6, -9 \cdot 10^6)$ at the two appended positions. The verified observation states that *some* zero-sum-pair structure is necessary for high-basin fixing rules at $d = 7$ — not necessarily on appended positions, but somewhere in the rule. Type A LOCK occurs precisely when F's nonzero digit arrangement in its sorted-descending form forbids any zero-sum-pair configuration from satisfying $K(F_{\text{padded}}) = F$.

The within-thread sls observation of §6.4–§6.5 sits inside this broader framework: among the four $\{7, 6, 4, 1\}$-thread members at $d = 7$, all admit zero-sum-pair rules (12 such rules each for 60714, 170460, 607140; 36 for 146070). The Type A vs Type B distinction does not separate them — they are all Type B-or-better — and the finer ordering observed in §6.5.1 reflects within-multiset basin structure not captured by the coarser Type A / Type B partition.

6.6.4 Type B locking: an open question

Type B LOCK is mechanistically distinct from Type A but lacks a clean structural characterization. Across the 6 Type B LOCK cases in the verified records (9,225, 52,605, 67,023, 170,460 from the two-zero stratum; 284,679, 175,086 from the held-out set), zero-sum-pair rules exist but their basins fall in the range $[0.057, 0.92]$ without reaching the 0.95 NEAR threshold. The cause appears to be a quantitative bound on basin achievable by the available zsp rules, but we do not characterize it further here.

We conjecture that the Type A vs Type B distinction extends to higher d-transitions ($d = 7 \to d = 8$ and beyond), with the Type A characterization remaining identical (no zsp rule fixing F at the target d) and the Type B boundary potentially shifting. Verification at $d = 7 \to d = 8$ is left to future work.

6.7 Context and open directions

The $\{7, 6, 4, 1\}$-thread exposes three phenomena worth noting for the broader research direction.

1. The classical Kaprekar recipe degenerates at odd digit lengths. The rule abc$\cdots$ − (reverse) has algebraic rank d at even d but rank $d - 1$ at odd d, because the middle coefficient $10^{(d-1)/2} - 10^{(d-1)/2} = 0$ vanishes identically. Universal fixed points of this rule therefore can only exist at even d (where 6174 sits at $d = 4$, and 631,764 — though not full-variable in our sense — sits at $d = 6$; cf. §3). At odd d, universal full-variable fps must come from non-classical rules. 60714 is the canonical such fp at $d = 5$.

2. 6174 and 60714 share digit content but realize it through different rules. The two fps live in the same digit multiset $\{7, 6, 4, 1, 0\}$ (with 6174 padded) but are fixed points of structurally different permutation pairs. 6174's native rule has sum-zero coefficient structure $(999, 90, -90, -999)$ that produces an algebraic four-of-a-kind escape class. 60714's native rule places its largest coefficient on the zero digit, eliminating the analogous escape class. The result: 60714 is universally convergent on $A_d \setminus E_d$ at every $d \geq 5$ with $E_d = \emptyset$ at $d = 5, 6$, while 6174 has a 45-input escape class persisting at every $d \geq 8$ under its native lifting.

3. 60714 is the uniquely complete case. Within the thread, 60714 is the only fixed point for which uniform universality at every d in its range is both observed empirically and proven rigorously (Theorem 5.2). 6174 is partially universal (NEAR everywhere from $d = 5$ onward) with a proven pattern; 60417 is dimension-locked at $d \geq 6$. The sum_locked_spans invariant provides an empirical within-thread ordering of these three behaviors (plus 1746 as a within-multiset control) but does not yet yield a uniform theorem analogous to Theorem 5.2 for fps other than 60714. Extending the lifting-theorem framework to 6174 — proving the monotone NEAR-convergence of basin to 1 across all $d \geq 5$ — is a natural open direction (§7).

4. The cycle-signature pattern at $d = 7$ is a consequence of Type A locking, not an independent classifier. The cycle-signature observation at $d_F = 6$ — that signature $(3, 3)$ uniformly produces dimension-locked rules — does not extend to signature $(3, 4)$ at $d = 7$ (Appendix D shows $(3, 4)$ produces both TRANS and LOCK). A refined empirical observation does hold at $d = 7$: among the 53 verified two-zero $d = 6$ fps, those whose F-fixing rule space is restricted to only the long-cycle signatures $\{(3, 4), (7,)\}$ uniformly LOCK (4 of 4: $F = 17{,}019, 37{,}017, 701{,}901, 707{,}130$). All four cases are also Type A LOCK by Observation 6.3, which suggests the cycle-signature restriction is a *consequence* of the more fundamental zsp absence rather than an independent classifier. The Type A characterization of §6.6 supersedes the cycle-signature pattern at $d = 7$.

6.8 Summary

Theorem 6.1 establishes the cross-dimensional pattern of 6174 at $d = 4, \ldots, 9$: strict universal at $d = 4$ (Kaprekar's 1949 result), algebraically obstructed at $d = 5$, dynamically obstructed at $d = 6$

(basin 0.969 below NEAR threshold), NEAR-universal at $d = 7, 8, 9$ with monotonically improving basin and a precisely characterized 45-input escape class at $d \geq 8$. The sum_locked_spans invariant (§6.4) organizes the four-fixed-point thread (Observation 6.2) as a within-thread empirical regularity: 1746 (sls = 4, NEAR), 60714 (sls = 5, strict universal), 60417 (sls = 7, dimension-locked beyond $d = 5$), and 6174 (sls = 8, monotone NEAR). This unification is explicitly labeled as empirical and within-thread; cross-multiset testing on the $\{2, 3, 5, 8\}$-thread shows the invariant is not a global predictor. 60714 is the uniquely complete case — the universal attractor that the classical Kaprekar recipe at $d = 5$ could not see, but which exists in the same digit multiset as 6174 once non-classical full-variable rules are admitted.

End of §6.

7 Open Questions and Conjectures

The results of §3–§6 establish a specific empirical and structural picture at digit lengths $d \leq 9$ for the multiset thread $\{7, 6, 4, 1\}$, with a full theorem (§5, Theorem 5.2) covering 60714 at every $d \geq 5$. Several natural questions extend beyond what we have proven; we collect them here.

7.1 Dimension-locking as a general phenomenon

Theorem 4.1 established that 32 of 33 universal full-variable fixed points at $d = 5$ are dimension-locked at $d = 5$ — no rank-6 rule at $d = 6$ is universal for them. The natural generalization is:

Conjecture 7.1 (effective rank bound at cross-dimensional universality). *Let F be a universal full-variable fixed point at native digit length d_F, and suppose K is a full-variable rule at some $d \neq d_F$ that is universal for F. Then $\mathrm{sv}_F(K) < d$ — that is, the effective rank of K at F is strictly less than the algebraic rank.*

Informally: when a fixed point appears at a non-native digit length, it always does so through a rule whose effective rank at F is strictly reduced — some of the rule's coefficients are "absorbed" by F's zero digits and do not contribute to the computation $K(F) = F$.

Status. Verified exhaustively at $d = 6$ for all 33 $d = 5$ fixed points (Theorem 4.1 and the associated enumeration). For 60714, Theorem 5.2's construction automatically satisfies the conjecture at every $d \geq 6$: the coefficient-preserving lifting has $\mathrm{sv}_F = 4 < d$ at every $d \geq 6$. For 6174, the liftings at $d = 7, 8, 9$ similarly satisfy $\mathrm{sv}_F \leq d - 3 < d$. The conjecture has not been tested at $d > 6$ for the other 32 dimension-locked $d = 5$ fixed points or for fps native at $d = 6$.

Formal verification of Conjecture 7.1 at $d \geq 7$ is open. Verifying it at a specific $d \geq 7$ for a specific fixed point F reduces to exhaustive enumeration over rank-d rules satisfying $K(F_{(d)}) = F$, checking that all such rules have $\mathrm{sv}_F < d$. The enumeration is tractable at $d = 7$ (several million rules per fixed point) and becomes computationally demanding at $d \geq 8$.

7.2 The structural invariant and dimensional fate

Observation 6.2 records the empirical unification of 60714, 60417, and 6174 through the sls invariant. The natural formal statement is:

Conjecture 7.2 (structural invariant controls dimensional fate). *For a universal full-variable fixed point F with native digit length d_F and native rule K_F, the cross-dimensional behavior of F at $d > d_F$ is controlled by* $\mathrm{sum\_locked\_spans}(K_F)$ *and the number of zero digits of F at d. Specifically:*

- *If* $\mathrm{sum\_locked\_spans}(K_F)$ *is small relative to the number of absorbing positions at d, a coefficient-preserving lifting exists that is universal for F at d.*
- *If the invariant is large relative to the absorbing capacity, either no lifting exists (algebraic obstruction) or liftings exist but no one is universal (dynamic obstruction).*

Status. Purely conjectural. The empirical pattern holds for 60714, 60417, 6174; whether it holds for arbitrary universal full-variable fixed points at $d_F = 4, 5, 6$ has not been systematically tested. A positive resolution would give a quantitative criterion for predicting cross-dimensional transcendence from the native rule alone, obviating the need for direct enumeration at each candidate d.

7.3 Other multiset clusters

Question 7.1. *Which digit multisets support universal full-variable fixed points at three or more consecutive native digit lengths?*

The $\{7, 6, 4, 1\}$ thread is the only such example in our classification at $d \leq 6$. The other $d = 4$ multiset $\{2, 3, 5, 8\}$ produces fps at $d = 4$ but not at $d = 5, 6$. Several $d = 5$ multisets have analogues at $d = 4$ or $d = 6$ but not at both.

Extending the search to $d \geq 7$ would require either a full classification at $d = 7$ (the $d = 7$ enumeration of 9,344,160 full-variable rules is computationally demanding; see Appendix D) or a more structural argument for which multisets can support cross-dimensional threads.

Question 7.2. *Is there an analogue of* 60714 *at higher native digit lengths — a fixed point F with native $d_F = 7$ or $d_F = 8$ that is universal at every $d \geq d_F$ via a coefficient-preserving lifting?*

Such fixed points would constitute a second (or third, etc.) dimension-transcendent attractor family. Observed data at $d_F = 6$ and $d_F = 7$ (Appendix D) contain several candidates that transcend to $d + 1$, but none has been proven to lift uniformly to every higher d as 60714 does.

7.4 Other bases

All results in this paper concern base 10. The analogous framework in base $b \geq 2$ would replace 10^{π_i} with b^{π_i} in the coefficient expansion of §2.2 and otherwise proceed identically.

Question 7.3. *For each base b, does there exist a universal full-variable fixed point at some native digit length that is universal at every $d \geq d_F$ via a coefficient-preserving lifting?*

At base 10, the answer is yes, demonstrated by 60714. For small bases ($b = 2, 3, \ldots$), the analogous classification would produce different specific fixed points, and the structural mechanism of coefficient-preserving lifting would apply with appropriate modifications. The related literature [Kay 2024, Peterson 2008] addresses the classical Kaprekar rule at other bases; our framework has not been systematically extended.

7.5 The 6174 pattern at $d \geq 10$

Theorem 6.1 establishes 6174's behavior at $d = 4, 5, \ldots, 9$. The pattern at $d = 8, 9$ (near-universal with 45-input escape class) is expected to continue at $d \geq 10$: the escape class structure is d-

independent (same multiset shape at each d), while the admissible input set grows, so the basin fraction asymptotes to 1.

Question 7.4. *Is* 6174 *strict-universal at any* $d \geq 10$*?*

Empirical extrapolation suggests no: at each $d \geq 8$, the 45-input escape class persists, and no coefficient-preserving lifting eliminates it. A proof (or disproof) of this pattern at some specific large d would complete the 6174 pattern to an arbitrary horizon. At the structural level, proving that the 45-input escape class is genuinely stable under all coefficient-preserving liftings is an algebraic question on the coefficient vectors' action on multisets of the form $(X, X, X, X, 0, \ldots, 0)$.

7.6 The depth of the escape class — the Basin Density Conjecture

Before stating the Basin Density Conjecture, we record a structural observation that places the escape class E_d in continuity with the classical Kaprekar phenomenon at $d = 3, 4$.

Observation 7.6.0 (continuity with the classical case). *Classical Kaprekar dynamics already exhibits an escape class at* $d = 3$ *and* $d = 4$*. At* $d = 3$*, the ten repdigits* $\{0, 111, 222, \ldots, 999\}$ *each map to* 0 *in one application of* K_0*, rather than to the nontrivial fixed point* 495*. At* $d = 4$*, the ten multiples of* 1111 *— namely* $\{0, 1111, 2222, \ldots, 9999\}$ *— each map to* 0 *in one application of* K_0*, rather than to* 6174*. In both cases, the escape class is precisely the set of inputs whose sorted-descending and sorted-ascending forms are identical, and these are exactly the inputs that the literature has conventionally treated as "inadmissible." The escape class* E_d *defined by Definition 5.2.1 for the lifted operator* $K^{(d)}$ *at* $d \geq 7$ *is the same phenomenon at higher digit lengths: a structurally characterized set of inputs whose orbit reaches the block-aligned multiset set* B_d *and then collapses to* 0*, rather than reaching the nontrivial fixed point* 60714*. The set* $B_d \cap A_d$ *that anchors the escape class plays a role precisely analogous to that of the repdigits at* $d = 3$ *and the multiples of* 1111 *at* $d = 4$*.*

What is new at $d \geq 7$ is therefore not the existence of an escape class — that is a generic feature of digit-permutation dynamics under any sum-zero rule, and the classical case at $d = 3, 4$ has always had one. What is new is the *explicit closed form* for $|E_d^{(1)}|$ given by Lemma 5.5, the proof of one-step collapse for inputs in $B_d \cap A_d$ (Lemma 5.1), and the structural conjecture below that the asymptotic basin density $|E_d|/|A_d| \to 0$ across the lifting family. The classical $d = 3, 4$ escape classes are small and structurally simple precisely because the digit space is small; the lifted-operator escape class at $d \geq 7$ is larger but is governed by an analogous block-alignment criterion. Conjecture 7.6 asks whether the escape class, while growing combinatorially with d, has vanishing density.

Lemma 5.5 gives a closed form for $|E_d^{(1)}|$ — the number of step-1 collapse inputs at digit length d. The full escape class E_d is the backward orbit of $B_d \cap A_d$, and at $d \leq 11$ exhaustive enumeration confirms that every orbit in E_d reaches 0 in at most 4 iterations. We name the asymptotic claim:

Conjecture 7.6 (Basin Density Conjecture). *The Basin Density Conjecture for the* 60714 *lifting family has two parts:*

1. ***(Density.)*** $|E_d|/|A_d| \to 0$ *as* $d \to \infty$*, i.e., the basin of* 60714 *at digit length* d *has asymptotic density* 1 *within admissible inputs.*
2. ***(Depth.)*** *There exists a uniform constant* T^* *such that every orbit in* E_d *reaches* 0 *in at most* T^* *iterations of* $K^{(d)}$*, independently of* d*.*

Combinatorially, $|E_d^{(1)}|$ grows polynomially in d (specifically as $O(d^{d_0-1})$ via Lemma 5.5) while $|A_d|$

grows as $O(d^9)$ with much larger constant; the basin fraction $1 - |E_d|/|A_d|$ should approach 1, supporting the Density part. For the Depth part, exhaustive enumeration at $d \le 11$ shows $T^* \le 4$ uniformly across the verified range, but no proof of a uniform bound for general d is provided here.

7.7 Beyond universal full-variable fixed points

The full-variable constraint $\mathrm{sv} = d$ excludes dimension-agnostic fixed points like 45, 495, 450 (§2.5). These fixed points exhibit their own cross-dimensional behavior through rank-reducing rules, which we have not addressed.

Question 7.5. *Is there a useful cross-dimensional theory for fixed points of rank-reducing rules* $(\mathrm{sv} < d)$*?*

The dimension-agnostic family is the simplest case: these fixed points persist across all digit lengths by construction. A more interesting question is whether there are non-full-variable rules whose cross-dimensional behavior mirrors 60714's full-variable universality — say, rules with $\mathrm{sv} = d - 1$ that produce universal fixed points at every d in some range. We leave this open.

7.8 Concluding remarks

The central result of this paper is Theorem 5.2: 60714 is a universal full-variable fixed point at every digit length $d \ge 5$, under an explicit coefficient-preserving lifting. This is the first explicitly constructed cross-dimensional persistence result in the generalized Kaprekar family. The supporting results — the classifications of §3, the uniqueness of §4, and the 6174 cross-dimensional pattern of §6 — place this theorem in a structural context that raises more questions than it answers. Conjectures 7.1 and 7.2 and Questions 7.1–7.5 and the Basin Density Conjecture (7.6) together outline a program for understanding cross-dimensional behavior in the generalized family: which fixed points transcend, under what structural conditions, and with what mechanism.

The methodology of the paper — exhaustive enumeration at each digit length, combined with structural proof on the cases the enumeration distinguishes — has been effective at $d \le 6$ for the full rule space and at $d \le 9$ for targeted verification of specific fps. Extending this methodology to $d \ge 7$ for the full rule space, or to $d \ge 10$ for 6174, is the natural next step.

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8 Acknowledgments

The mathematical content of this paper — the framing of the organizing question, the classifications at $d = 3, 4, 5, 6$, the cross-dimensional cross-check, the coefficient-preserving lifting framework, the proof of Theorem 5.2 (including the structural arguments of Lemmas 5.1, 5.2, 5.3, 5.4 and the support lemmas of Appendix C), the $\{7, 6, 4, 1\}$-thread analysis, and the open-question program of §7 — was developed by the author, who is solely responsible for the paper's content, including any remaining errors.

The development of the paper used substantial computer assistance, in the spirit of the increasing role of computational tools in number-theoretic research. Verification scripts were written and executed for every numerical claim in the paper — classifications, basin counts, ladder-extension data, escape-class enumerations. Structural arguments (most notably Lemma C.10's case enumeration and the $d \ge 15$ algebraic argument in Appendix C.4) were tested against worked examples and

edge cases, a process that identified and corrected several defects during development — including a parenthetical algebraic error in an earlier version of Lemma C.9, an enumeration-count error in an earlier version of Lemma C.10, and an off-by-one error in a closed-form formula in supplementary material. Prose was edited iteratively. The cross-reference of the escape-class size sequence to OEIS A000582 was identified through assisted literature search.

These computer-assisted activities used AI tools (Anthropic's Claude, model versions Claude Opus 4.6 and 4.7, accessed during 2025 and 2026) alongside conventional scripting in Python. The full development record, the verification scripts, and the iterative review history are publicly available at https://github.com/clayelmore/Kaprekar-60714.

Theorems, lemmas, propositions, conjectures, proofs, and computational claims are the author's; computational tools were used as instruments for verification, drafting, and review, not as authors.

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9 Appendix A. Classification Tables at $d = 3, 4, 5, 6$

This appendix provides the complete enumerated classifications at digit lengths $d = 3, 4, 5, 6$. Each section lists all universal full-variable fixed points at the given d along with representative data for one universal rule per fixed point. Sign-flipped duplicates are not listed separately (see Proposition 2.2).

The classifications were computed by exhaustive enumeration in Python. Runtime on commodity hardware: negligible at $d = 3, 4$; ~5 seconds at $d = 5$; ~5–10 minutes at $d = 6$. Source code for the enumerations is available in the supplementary materials.

9.1 Classification at $d = 3$

Full-variable rules enumerated: $3! \cdot D_3 = 6 \cdot 2 = 12$. **Admissible inputs enumerated:** 720 (non-repdigit, non-near-repdigit 3-digit integers). **Universal full-variable fixed points:** 0.

No full-variable rule at $d = 3$ is universal for any fixed point. For every rule, iteration from at least some admissible input enters a cycle or a non-universal fixed point.

Note on the dimension-agnostic fixed point 495. The classical rule K_0 at $d = 3$ is universal for 495, but has algebraic rank 2 (the middle-position coefficient vanishes; see §1.1). 495 is thus an sv$= 2$ fixed point, not an sv$= 3$ fixed point, and is outside the scope of the full-variable classification.

9.2 Classification at $d = 4$

Full-variable rules enumerated: $4! \cdot D_4 = 24 \cdot 9 = 216$. **Admissible inputs enumerated:** 615 digit multisets (9,630 padded four-digit strings). **Universal full-variable fixed points:** 4. **Universal full-variable rules (total, counting sign-flip pairs):** 8.

Fixed point	Digit multiset	Sample π	Sample σ	Sample coefficient vector
1746	$\{1, 4, 6, 7\}$	$(0, 3, 2, 1)$	$(1, 2, 3, 0)$	$(-9, 900, -900, 9)$
2538	$\{2, 3, 5, 8\}$	$(1, 0, 3, 2)$	$(2, 3, 0, 1)$	$(-90, -999, 999, 90)$
5382	$\{2, 3, 5, 8\}$	$(2, 1, 0, 3)$	$(3, 0, 1, 2)$	$(-900, 9, -9, 900)$

Fixed point	Digit multiset	Sample π	Sample σ	Sample coefficient vector
6174	$\{1, 4, 6, 7\}$	$(0, 1, 2, 3)$	$(3, 2, 1, 0)$	$(-999, -90, 90, 999)$

Note on 6174**.** One of the two universal rules for 6174 is the classical Kaprekar rule K_0 (descending minus ascending), recovering Kaprekar's original 1949 result within our classification.

Note on anagram clusters. The four fps fall into two anagram pairs: $\{1746, 6174\}$ (multiset $\{1, 4, 6, 7\}$) and $\{2538, 5382\}$ (multiset $\{2, 3, 5, 8\}$). In each cluster, the two fps sum to 7920, a specific instance of reverse-pair identity at $d = 4$.

9.3 Classification at $d = 5$

Full-variable rules enumerated: $5! \cdot D_5 = 120 \cdot 44 = 5{,}280$. **Admissible inputs enumerated:** 99,540. **Universal full-variable fixed points:** 33. **Universal full-variable rules (total, counting sign-flip pairs):** 66.

The 33 fixed points grouped by digit multiset:

Multiset	Fixed points	Cluster size
$\{0, 0, 0, 4, 5\}$	54	1
$\{0, 3, 3, 5, 7\}$	3753	1
$\{0, 1, 4, 6, 7\}$	60714, 60417	2
$\{2, 3, 5, 8, 9\}$	28539, 53928, 58239	3
$\{1, 2, 3, 4, 8\}$	18342, 21834, 24183, 41832, 42183	5
$\{1, 5, 6, 7, 8\}$	16578, 16758, 17685, 65781, 67581	5
$\{3, 4, 5, 7, 8\}$	37584, 37854, 38754, 43758, 43785, 43875	6
$\{1, 2, 4, 5, 6\}$	12456, 14562, 15642, 16524, 21456, 24156, 24561, 41562, 45612, 45621	10

The complete list of 33 fixed points in ascending order:

$$54,\ 3753,\ 12456,\ 14562,\ 15642,\ 16524,\ 16578,\ 16758,\ 17685,\ 18342,$$
$$21456,\ 21834,\ 24156,\ 24183,\ 24561,\ 28539,\ 37584,\ 37854,\ 38754,\ 41562,$$
$$41832,\ 42183,\ 43758,\ 43785,\ 43875,\ 45612,\ 45621,\ 53928,\ 58239,\ 60417,$$
$$60714,\ 65781,\ 67581.$$

Representative rules at the $\{0, 1, 4, 6, 7\}$ **cluster** (the $\{7, 6, 4, 1\}$-thread of §6):

Fixed point	π	σ
60714	$(4, 1, 2, 3, 0)$	$(2, 0, 1, 4, 3)$
60417	$(4, 1, 0, 3, 2)$	$(0, 2, 1, 4, 3)$

(Coefficient vectors are recoverable as $c_i = 10^{\pi_i} - 10^{\sigma_i}$: $(9900, 9, 90, -9000, -999)$ for 60714 and $(9999, -90, -9, -9000, -900)$ for 60417.)

Note on 54. The fixed point 54 appears in the $d = 5$ full-variable classification under the rule $\pi = (1, 2, 4, 3, 0)$, $\sigma = (2, 0, 3, 4, 1)$ with coefficient vector $(-90, 99, 9000, -9000, -9)$. This rule is algebraically full-variable at $d = 5$ (sv $= 5$), but its effective rank at $F = 54$ is only 2 — three of the five coefficients land on zero digits of F's sorted-descending form $(5, 4, 0, 0, 0)$ and do not contribute to $K(F) = F$. Thus $\mathrm{sv}_F(K_{54}) = 2 < 5 = \mathrm{sv}(K_{54})$. This is structurally analogous to the 495 phenomenon at $d = 3$ (§1.1), occurring at higher digit length within a genuinely full-variable rule.

Note on 3753. Similarly, 3753 has sorted-descending form $(7, 5, 3, 3, 0)$ at $d = 5$ with one zero digit. Its effective rank under its native rule is $\mathrm{sv}_F = 4 < 5$.

Both 54 and 3753 are dimension-locked at $d = 5 \to d = 6$: the cross-check of §4 finds no universal full-variable rule for either at $d = 6$.

9.4 Classification at $d = 6$

Full-variable rules enumerated: $6! \cdot D_6 = 720 \cdot 265 = 190{,}800$. **Admissible inputs enumerated:** 4,905 digit multisets (999,450 padded six-digit strings). **Universal full-variable fixed points:** 506. **Universal full-variable rules (total, counting sign-flip pairs):** 1,174.

9.5 Summary statistics

The 506 fixed points distribute by zero-digit count:

Zero-digit count	Fixed-point count
0	205
1	240
2	53
3	8

And by digit sum (every universal fp at $d = 6$ has digit sum divisible by 9):

Digit sum	Fixed-point count
9	8
18	156
27	244
36	96
45	2

9.6 Distinguished fixed points

The following $d = 6$ universal full-variable fixed points are singled out elsewhere in the paper:

Fixed point	Digit multiset	Note
549945	$\{4, 4, 5, 5, 9, 9\}$	zero-zero fp; algebraically obstructed at $d = 6 \to d = 7$
60714	$\{0, 0, 1, 4, 6, 7\}$	central result; cross-dimensional fixed point at every $d \geq 5$, strict-universal at $d = 5, 6$ (Theorem 5.2)
146070	$\{0, 0, 1, 4, 6, 7\}$	in the $\{7, 6, 4, 1\}$-thread; native at $d = 6$; transcendent to $d = 7$
170460	$\{0, 0, 1, 4, 6, 7\}$	in the $\{7, 6, 4, 1\}$-thread; native at $d = 6$
607140	$\{0, 0, 1, 4, 6, 7\}$	in the $\{7, 6, 4, 1\}$-thread; native at $d = 6$; transcendent to $d = 7$
631764	$\{1, 3, 4, 6, 6, 7\}$	attractor at $d = 6$ under classical rule with basin 0.0625 [Dahl 2026]

9.7 Fixed points with two or more zero digits

The complete list of 506 universal full-variable fixed points at $d = 6$ is lengthy; we provide it in the supplementary materials as a plain-text file (`d6_fps.txt`). The enumeration is fully reproducible from the source code.

Of particular relevance to §5 (the 60714 theorem) and §6 (the $\{7, 6, 4, 1\}$-thread) are the fps with at least two zero digits, since these are the strata where dimension-transcendent behavior empirically concentrates (cf. Conjecture 7.2).

The 8 fps with 3 zero digits. All eight share digit multiset $\{5, 2, 2, 0, 0, 0\}$ and are distinct integer arrangements of the same digits:

$$252, \quad 2520, \quad 20025, \quad 25200, \quad 200025, \quad 200250, \quad 250200, \quad 252000.$$

That all eight 3-zero fps belong to a single multiset is a notable structural fact — at higher zero-count strata, the multiset diversity drops sharply.

The 53 fps with 2 zero digits, grouped by digit multiset:

Multiset	Cluster size	Sample fps
$\{9, 9, 8, 1, 0, 0\}$	32	$8919, 8991, 9189, 10899, 80919, \ldots$
$\{5, 5, 4, 4, 0, 0\}$	7	$4545, 44505, 54450, 445005, \ldots$

Multiset	Cluster size	Sample fps
$\{9,7,1,1,0,0\}$	5	$7191, 17019, 700191, 701901, \ldots$
$\{7,6,4,1,0,0\}$	4	$60714, 146070, 170460, 607140$
$\{7,7,3,1,0,0\}$	2	$37017, 707130$
$\{9,5,2,2,0,0\}$	1	9225
$\{7,6,3,2,0,0\}$	1	67023
$\{6,5,5,2,0,0\}$	1	52605

The complete sorted list of all 53 fps with 2 zero digits:

$$4545, 7191, 8919, 8991, 9189, 9225, 10899, 17019, 37017, 44505, 52605, 54450,$$
$$60714, 67023, 80919, 80991, 81099, 89019, 90819, 90918, 91809, 98091, 100899,$$
$$108099, 109809, 146070, 170460, 180909, 189009, 190089, 190809, 445005,$$
$$445050, 504450, 544500, 607140, 700191, 701901, 707130, 719100, 809091,$$
$$809109, 810909, 890091, 900891, 901890, 908091, 908109, 908190, 908901,$$
$$909081, 910089, 910809.$$

The $\{7,6,4,1,0,0\}$ multiset cluster. The four fps in this cluster — $60714, 146070, 170460, 607140$ — share the multiset structure of the central object of §5 (60714 padded to $d = 6$). They are precisely the universal full-variable fps in this multiset at $d = 6$, and the cross-dimensional behavior of all four is documented at §6 and in Appendix D.

Note on the unique 4-zero fp. §A.4.1 records one universal full-variable fp at $d = 6$ with 4 zero digits. Its identity is recorded in `d6_fps.txt` in the supplementary materials but is not analyzed individually in this paper.

Fixed points with 0 or 1 zero digit. The remaining 445 fps (205 with 0 zeros, 240 with 1 zero) are listed in the supplementary file and are not analyzed individually here. Cross-dimensional behavior at $d = 6 \to d = 7$ for fps with fewer than 2 zeros is empirically rare (cf. Observation D.1).

9.8 Cross-reference to §4 and §6

The 33 fixed points at $d = 5$ of §A.3 are the subject of §4's cross-check: the exhaustive computation of which are dimension-locked at $d = 5 \to d = 6$. The result (17 algebraic obstruction, 15 dynamic obstruction, 1 universal lifting to $d = 6$) is proven at that section.

The 8 fixed points with 3 zero digits at $d = 6$ of §A.4.1 are the subject of §6's transcendent-fp analysis at $d = 6 \to d = 7$ (empirical observations recorded in Appendix D).

End of Appendix A.

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10 Appendix B. Coefficient-preserving ladder for $F = 60714$

For each digit length $d \geq 5$, the following tables give the permutation-pair rule $K^{(d)}$ under which $F = 60714$ is a universal fixed point, as constructed in §5.

Conventions. At digit length d, sorted-descending digit positions are labeled $x_0, x_1, \ldots, x_{d-1}$ with $x_0 \geq x_1 \geq \cdots \geq x_{d-1}$. We assign letters $a, b, c, d, e, f, g, \ldots$ to these positions in order, so $a = x_0$, $b = x_1$, etc. The rule $K = \pi \cdot x - \sigma \cdot x$ is written as two letter strings: reading each string left-to-right, the k-th letter indicates which sorted-descending position contributes the digit at place 10^{d-1-k} of the resulting integer. The coefficient vector is $c_i = 10^{\pi_i} - 10^{\sigma_i}$ for $i = 0, 1, \ldots, d-1$.

Fixed-point verification. At every d, the assignment $a = 7, b = 6, c = 4, d = 1$, and every subsequent letter $= 0$ corresponds to $F = 60714$'s sorted-descending form (padded to d digits). Under this assignment, $\pi \cdot x$ evaluates to 71460 and $\sigma \cdot x$ evaluates to 10746, so $K(F) = |71460 - 10746| = 60714$ at every d.

10.1 Table B.1. Ladder rules through $d = 20$

(Per the fixed-point verification above, every row in this table gives $\pi \cdot F = 71460$, $\sigma \cdot F = 10746$, and $K(F) = 60714$. We omit those three columns to keep the letter strings legible at high d.)

d	Ladder	π (letters)	σ (letters)
5	odd (native)	`adcbe`	`deacb`
6	even (split)	`eadcbf`	`fdeacb`
7	odd	`fgadcbe`	`gfdeacb`
8	even	`hgeadcbf`	`ghfdeacb`
9	odd	`hifgadcbe`	`ihgfdeacb`
10	even	`jihgeadcbf`	`ijghfdeacb`
11	odd	`jkhifgadcbe`	`kjihgfdeacb`
12	even	`lkjihgeadcbf`	`klijghfdeacb`
13	odd	`lmjkhifgadcbe`	`mlkjihgfdeacb`
14	even	`nmlkjihgeadcbf`	`mnklijghfdeacb`
15	odd	`nolmjkhifgadcbe`	`onmlkjihgfdeacb`
16	even	`ponmlkjihgeadcbf`	`opmnklijghfdeacb`
17	odd	`pqnolmjkhifgadcbe`	`qponmlkjihgfdeacb`
18	even	`rqponmlkjihgeadcbf`	`qropmnklijghfdeacb`
19	odd	`rspqnolmjkhifgadcbe`	`srqponmlkjihgfdeacb`
20	even	`tsrqponmlkjihgeadcbf`	`stqropmnklijghfdeacb`

10.2 Table B.2. Coefficient vectors through $d = 12$

The coefficient vectors are the same data as the letter strings in Table B.1, presented numerically. The structure is uniform along each ladder: every rule begins with the four locked coefficients $(9900, 9, 90, -9000)$ at F's nonzero-digit positions, and every higher-d rule extends the previous one by appending one zero-sum pair.

The first five (odd ladder) or six (even ladder) coefficients are:

Ladder	Locked + root coefficients
odd, $d = 5$	$(9900,\ 9,\ 90,\ -9000,\ -999)$
even, $d = 6$	$(9900,\ 9,\ 90,\ -9000,\ 99000,\ -99999)$

For each subsequent step along a ladder, the rule at d extends the rule at $d-2$ by appending the zero-sum pair $(\pm 9 \cdot 10^{d-2},\ \mp 9 \cdot 10^{d-2})$. Concretely:

d	Ladder	Appended pair (c_{d-2}, c_{d-1})
7	odd	$(9 \cdot 10^5,\ -9 \cdot 10^5) =$ $(900\,000,\ -900\,000)$
8	even	$(-9 \cdot 10^6,\ 9 \cdot 10^6) =$ $(-9\,000\,000,\ 9\,000\,000)$
9	odd	$(9 \cdot 10^7,\ -9 \cdot 10^7) =$ $(90\,000\,000,\ -90\,000\,000)$
10	even	$(-9 \cdot 10^8,\ 9 \cdot 10^8) =$ $(-900\,000\,000,\ 900\,000\,000)$
11	odd	$(9 \cdot 10^9,\ -9 \cdot 10^9) =$ $(9 \cdot 10^9,\ -9 \cdot 10^9)$
12	even	$(-9 \cdot 10^{10},\ 9 \cdot 10^{10}) =$ $(-9 \cdot 10^{10},\ 9 \cdot 10^{10})$

The full coefficient vector at any d is the locked-plus-root prefix followed by the chain of appended pairs from $d-2$ down to the ladder root. The construction is stated formally in §B.3.

10.3 The construction, stated formally

The tables above are the output of a simple recursive recipe that generalizes to every $d \geq 5$:

Base cases.

Odd ladder root (at $d = 5$): $\pi = (4, 1, 2, 3, 0)$, $\sigma = (2, 0, 1, 4, 3)$, with coefficient vector

$$c^{(5)} = (9900, 9, 90, -9000, -999).$$

Even ladder root (at $d = 6$): $\pi = (4, 1, 2, 3, 5, 0)$, $\sigma = (2, 0, 1, 4, 3, 5)$, with coefficient vector

$$c^{(6)} = (9900, 9, 90, -9000, 99000, -99999).$$

Inductive step. Given the rule at d on a ladder, the rule at $d+2$ on the same ladder is obtained by appending two new positions:

$$\pi_d = \begin{cases} d+1 & \text{(odd ladder)} \\ d & \text{(even ladder)} \end{cases}, \quad \pi_{d+1} = \begin{cases} d & \text{(odd ladder)} \\ d+1 & \text{(even ladder)} \end{cases},$$

$$\sigma_d = \begin{cases} d & \text{(odd ladder)} \\ d+1 & \text{(even ladder)} \end{cases}, \quad \sigma_{d+1} = \begin{cases} d+1 & \text{(odd ladder)} \\ d & \text{(even ladder)} \end{cases}.$$

In both cases, the two appended coefficients $c_d = 10^{\pi_d} - 10^{\sigma_d}$ and $c_{d+1} = 10^{\pi_{d+1}} - 10^{\sigma_{d+1}}$ satisfy $c_d + c_{d+1} = 0$ — they form a **zero-sum pair**. The only difference between ladders is the sign convention on the new pair.

10.4 Table B.3. Demonstration at $d = 100$

To show the recipe is not merely finite-table-bound but scales mechanically, we give the rule at $d = 100$ on the even ladder.

Structure of the coefficient vector at $d = 100$.

- Positions 0 through 3: $(9900,\ 9,\ 90,\ -9000)$. These are the coefficients at F's four nonzero-digit positions, preserved verbatim from the native rule at $d = 5$. (Locked nonzero coefficients.)
- Positions $4, 5$: $(99000,\ -99999)$. These are the even-ladder "split" coefficients fixed at the even root ($d = 6$).
- Positions $6, 7, 8, \ldots, 99$: 47 zero-sum pairs, one appended at each even-ladder step from $d = 8$ to $d = 100$.

The k-th appended pair (for $k = 1, 2, \ldots, 47$) consists of

$$c_{2k+4} = -10^{2k+4} \cdot 9 + (\text{higher-order terms}) = -\underbrace{\overbrace{9 \cdot 10^{2k+4}}^{\text{negative}}}_{\text{(see table)}}, \qquad c_{2k+5} = +9 \cdot 10^{2k+4}.$$

Concretely, the first four appended pairs at $d = 10, 12, 14, 16$ are:

Pair index k	Positions	(c_i, c_{i+1})
1	(6, 7)	$(-9 \times 10^6,\ +9 \times 10^6)$
2	(8, 9)	$(-9 \times 10^8,\ +9 \times 10^8)$
3	(10, 11)	$(-9 \times 10^{10},\ +9 \times 10^{10})$
4	(12, 13)	$(-9 \times 10^{12},\ +9 \times 10^{12})$
$\vdots$	$\vdots$	$\vdots$
47	(98, 99)	$(-9 \times 10^{98},\ +9 \times 10^{98})$

Fixed-point verification at $d = 100$. The digits of $F = 60714$ padded to 100 digits, sorted in descending order, are $a = 7$, $b = 6$, $c = 4$, $d = 1$, and 96 trailing zeros. Under this assignment, every appended coefficient c_i for $i \geq 4$ multiplies a zero digit and contributes nothing to $K(F)$. The computation reduces to

$$K(F) = |(9900)(7) + (9)(6) + (90)(4) + (-9000)(1)| = |\,69300 + 54 + 360 - 9000\,| = 60714.$$

This is the same arithmetic at every $d \geq 5$. **The coefficient-preserving lifting makes the fixed-point equation trivial at every d** because the structural content of the lifting lives entirely in the zero-digit positions, which absorb the new coefficients without affecting $K(F)$.

The nontrivial content of Theorem 3 is not that $K(F) = F$ at every d — that is automatic from the construction. The nontrivial content is that every admissible orbit outside the escape class E_d at every $d \geq 5$ reaches F, which is the work done by Lemmas 5.1 and 5.2.

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11 Appendix C. Support Lemmas for the Proof of Theorem 5.2

This appendix provides the technical support lemmas referenced in §5. The organization mirrors the proof of Theorem 5.2: §C.1 fixes setup, §C.2 proves core non-negativity (Lemma 5.3), §C.3 proves core upper bound (Lemma 5.4), §C.4 proves the reaching-time bound on the odd ladder, §C.5 addresses the even ladder, and §C.6 establishes the admissible-projection lemma.

11.1 Setup

Throughout, $(x_0, x_1, \ldots, x_{d-1})$ denotes a sorted-descending digit sequence with $x_0 \geq x_1 \geq \cdots \geq x_{d-1}$ and each $x_i \in \{0, 1, \ldots, 9\}$. The **native core contribution** (Definition 5.4) is

$$\mathrm{core}(x) \;=\; 9900\, x_0 + 9\, x_1 + 90\, x_2 - 9000\, x_3 - 999\, x_4.$$

11.2 Core non-negativity (Proof of Lemma 5.3)

Lemma 5.3 (restated). *For every sorted-descending sequence* $(x_0, x_1, x_2, x_3, x_4)$*,* $\mathrm{core}(x) \geq 0$.

Proof. We proceed by case analysis on the values of x_3 and x_4.

Case 1: $x_3 = x_4 = 0$. Then $\mathrm{core}(x) = 9900x_0 + 9x_1 + 90x_2 \geq 0$, with equality iff $x_0 = x_1 = x_2 = 0$.

Case 2: $x_3 \geq 1$, $x_4 = 0$. By sorted-descending, $x_0 \geq x_3 \geq 1$. Writing $\mathrm{core}(x) = 9000(x_0 - x_3) + 900x_0 + 9x_1 + 90x_2$, each term is non-negative, so $\mathrm{core}(x) \geq 0$.

Case 3: $x_3 \geq 1$ and $x_4 \geq 1$. We use $x_2 \geq x_3$ and $x_1 \geq x_4$:

$$\mathrm{core}(x) = 9900x_0 + 9x_1 + 90x_2 - 9000x_3 - 999x_4.$$

Replace $90x_2 \geq 90x_3$ (valid since $x_2 \geq x_3$):

$$\mathrm{core}(x) \geq 9900x_0 + 9x_1 + 90x_3 - 9000x_3 - 999x_4 = 9900x_0 + 9x_1 - 8910x_3 - 999x_4.$$

Replace $9x_1 \geq 9x_4$:

$$\mathrm{core}(x) \geq 9900x_0 - 8910x_3 - 999x_4 + 9x_4 = 9900x_0 - 8910x_3 - 990x_4.$$

Replace $x_4 \leq x_3$:

$$\text{core}(x) \geq 9900x_0 - 8910x_3 - 990x_3 = 9900x_0 - 9900x_3 = 9900(x_0 - x_3) \geq 0,$$

using $x_0 \geq x_3$ in the last step.

All three cases cover the admissible possibilities (Case 4 with $x_3 = 0$ and $x_4 \geq 1$ is impossible by sorted-descending). In each, $\text{core}(x) \geq 0$. $\square$

Remark C.1. The minimum $\text{core}(x) = 0$ is achieved at $x = (0, 0, 0, 0, 0)$ (repdigit zero). Direct enumeration over all 2,002 sorted-descending sequences in $\{0, \ldots, 9\}^5$ confirms the minimum is exactly 0.

11.3 Core upper bound (Proof of Lemma 5.4)

Lemma 5.4 (restated). *For every sorted-descending sequence* $(x_0, x_1, x_2, x_3, x_4)$*,* $\text{core}(x) \leq 89{,}991 < 10^5$.

Proof. The negative terms $-9000x_3 - 999x_4$ contribute at most 0 (when $x_3 = x_4 = 0$), so:

$$\text{core}(x) \leq 9900x_0 + 9x_1 + 90x_2 \leq 9900 \cdot 9 + 9 \cdot 9 + 90 \cdot 9 = 89{,}100 + 81 + 810 = 89{,}991.$$

The maximum is achieved at $(x_0, x_1, x_2, x_3, x_4) = (9, 9, 9, 0, 0)$. $\square$

Corollary C.2 (computational verification). *Direct enumeration over all* 2,002 *sorted-descending sequences* $(x_0, \ldots, x_4) \in \{0, \ldots, 9\}^5$ *confirms* $\text{core}(x) \in [0, 89{,}991]$ *with minimum* 0 *at* $(0, 0, 0, 0, 0)$ *and maximum* 89,991 *at* $(9, 9, 9, 0, 0)$*. Runtime: milliseconds.*

11.4 The T_d reaching-time bound (odd ladder)

The main technical claim underlying Proposition 5.3 rests on a **block-structure decomposition** of the lifted rule's output.

11.5 Block-structure decomposition

At odd digit length $d \geq 7$, the odd-ladder rule $K_{60714}^{(d)}$ has coefficient vector

$$c^{(d)} = \underbrace{(9900, 9, 90, -9000, -999)}_{\text{native, positions } 0..4} \parallel \underbrace{(+p_1, -p_1)}_{\text{pair 1,positions 5,6}} \parallel \cdots \parallel \underbrace{(+p_M, -p_M)}_{\text{pair } M\text{,positions } d-2, d-1},$$

where $M = (d-5)/2$ and $p_k = 9 \cdot 10^{3+2k}$ is the pair-k magnitude. Pair k occupies positions $(3 + 2k, 4 + 2k)$.

Pair contribution. For input $(x_0, \ldots, x_{d-1})$, each pair k contributes

$$p_k \cdot x_{3+2k} - p_k \cdot x_{4+2k} = p_k \, \delta_k, \qquad \text{where } \delta_k := x_{3+2k} - x_{4+2k} \in \{0, 1, \ldots, 9\}.$$

By the sorted-descending constraint, each $\delta_k \geq 0$.

Lemma C.6 (cliff-sequence bound). *For any sorted-descending sequence* $(x_0, \ldots, x_{d-1})$ *with each* $x_i \in \{0, \ldots, 9\}$,

$$\sum_{k=1}^{M} \delta_k \;=\; \sum_{k=1}^{M} (x_{3+2k} - x_{4+2k}) \;\le\; 9.$$

Proof. Each δ_k is a non-negative adjacent-position difference within the sorted-descending sequence. The sum is a subset of the telescoping total $\sum_{i=3}^{d-2}(x_i - x_{i+1}) = x_3 - x_{d-1}$. Since all differences are non-negative and $x_3 - x_{d-1} \le 9$, the selected subsum is bounded by the full telescoping:

$$\sum_{k=1}^{M} \delta_k \le x_3 - x_{d-1} \le 9.$$

□

11.6 Non-negativity of $K^{(d)}(x)$

Lemma C.7. *For every sorted-descending $(x_0, \dots, x_{d-1})$ and every odd $d \ge 7$,*

$$K^{(d)}(x) \;=\; \mathrm{core}(x) + \sum_{k=1}^{M} p_k\, \delta_k.$$

In particular, $K^{(d)}(x) \ge 0$, so the absolute value operation in the rule definition is vacuous on sorted-descending inputs.

Proof. Direct substitution into the expanded rule gives

$$K^{(d)}(x) \;=\; \left| \sum_{i=0}^{d-1} c_i\, x_i \right| \;=\; \left| \mathrm{core}(x) + \sum_{k=1}^{M} p_k\, \delta_k \right|.$$

By Lemma 5.3, $\mathrm{core}(x) \ge 0$. Each $p_k\, \delta_k \ge 0$ (since $p_k > 0$ and $\delta_k \ge 0$). Hence the argument is non-negative and the absolute-value bars can be dropped. □

11.7 The reaching-time theorem

Lemma C.3 (one-step T_d closure on the odd ladder at $d \ge 15$). *Let $d \ge 15$ be odd. For every admissible input $n \in A_d$ with sorted-descending form $(x_0, \dots, x_{d-1})$,*

$$K^{(d)}_{60714}(n) \in T_d.$$

Equivalently, the padded d-digit representation of $K^{(d)}(n)$ has at least two zero digits.

Proof. By Lemma C.7,

$$K^{(d)}(x) \;=\; \mathrm{core}(x) + \sum_{k=1}^{M} 9 \cdot 10^{3+2k}\, \delta_k.$$

Decimal-block structure. Each term $9 \cdot 10^{3+2k}\delta_k$ occupies decimal positions $[3+2k, 4+2k]$ as a two-digit value. Writing $9\delta_k = 10u_k + v_k$ for $u_k, v_k \in \{0, \dots, 9\}$:

$$9 \cdot 10^{3+2k}\delta_k = u_k \cdot 10^{4+2k} + v_k \cdot 10^{3+2k}.$$

The digit pairs (u_k, v_k) for each δ_k:

δ_k	$9\delta_k$	(u_k, v_k)	Zeros
0	0	$(0,0)$	2
1	9	$(0,9)$	1
2	18	$(1,8)$	0
3	27	$(2,7)$	0
4	36	$(3,6)$	0
5	45	$(4,5)$	0
6	54	$(5,4)$	0
7	63	$(6,3)$	0
8	72	$(7,2)$	0
9	81	$(8,1)$	0

Disjoint blocks. Pair k's block occupies decimals $[3+2k, 4+2k]$. Since the block value $9\delta_k \le 81 < 100$, it fits entirely within two decimal positions and does not propagate carries to higher blocks. Similarly, $\mathrm{core}(x) \le 89{,}991 < 10^5$ (Lemma 5.4) fits within decimals $[0, 4]$ with at most one carry into decimal 5. That carry combines additively with pair 1's block at positions $[5, 6]$: the combined value is at most $u_1 \cdot 10 + v_1 + 1 = 9\delta_1 + 1 \le 82$, still a two-digit value, so no further carry to decimal 7. Blocks for $k \ge 2$ retain their (u_k, v_k) form exactly.

Counting zeros in the block region. Define

$$Z_0 := \bigl|\{k : \delta_k = 0\}\bigr|, \qquad Z_1 := \bigl|\{k : \delta_k = 1\}\bigr|, \qquad Z_{2+} := \bigl|\{k : \delta_k \ge 2\}\bigr|.$$

Then $Z_0 + Z_1 + Z_{2+} = M$, and the number of zero digits contributed by blocks is at least $2Z_0 + Z_1$ (each $\delta_k = 0$ block contributes 2 zeros; each $\delta_k = 1$ block contributes 1 zero from $u_k = 0$).

From Lemma C.6, $\sum_k \delta_k \le 9$. Since $\delta_k \ge 2$ for $k \in Z_{2+}$:

$$2Z_{2+} + Z_1 \le \sum_k \delta_k \le 9.$$

Therefore:

$$2Z_0 + Z_1 \;=\; 2(M - Z_1 - Z_{2+}) + Z_1 \;=\; 2M - Z_1 - 2Z_{2+} \;\ge\; 2M - 9.$$

At odd $d \ge 17$: $M \ge 6$, so $2Z_0 + Z_1 \ge 3$. This alone gives at least 3 zero digits in the block region, more than enough for $K^{(d)}(x) \in T_d$.

At $d = 15$ (odd): $M = 5$, so $2Z_0 + Z_1 \geq 1$. This gives at least one zero digit from the block region, but the second zero requires a refined analysis. We complete the argument by partitioning according to the value of x_4.

Lemma C.7 (sharpened cliff-sum bound for $d = 15$). *For sorted-descending $(x_0, \ldots, x_{14})$ with $x_i \in \{0, \ldots, 9\}$,*

$$\sum_{k=1}^{5} \delta_k \;\leq\; x_4.$$

Proof. Each $\delta_k = x_{3+2k} - x_{4+2k}$ for $k = 1, \ldots, 5$ is a non-negative adjacent-position difference among positions $5, 6, \ldots, 14$. The full telescoping over these positions is

$$\sum_{i=5}^{13}(x_i - x_{i+1}) = x_5 - x_{14}.$$

Our sum $\sum_k \delta_k$ picks out the five differences $(x_5 - x_6), (x_7 - x_8), (x_9 - x_{10}), (x_{11} - x_{12}), (x_{13} - x_{14})$, a strict subset of these nine. Since every omitted difference is non-negative,

$$\sum_{k=1}^{5} \delta_k \;\leq\; x_5 - x_{14} \;\leq\; x_5 \;\leq\; x_4. \qquad \square$$

Lemma C.8 (core-bound when $x_4 \in \{8, 9\}$). *If $x_4 \geq 8$ in sorted-descending $(x_0, \ldots, x_4)$, then* $\mathrm{core}(x) < 10^4$.

Proof. By sorted-descending, $x_4 \geq 8$ forces $x_0, x_1, x_2, x_3 \in \{8, 9\}$. We compute:

$$\mathrm{core}(x) = 9900x_0 + 9x_1 + 90x_2 - 9000x_3 - 999x_4 = 9000(x_0 - x_3) + 900x_0 + 9x_1 + 90x_2 - 999x_4.$$

Since $x_0, x_3 \in \{8, 9\}$ with $x_0 \geq x_3$, we have $x_0 - x_3 \in \{0, 1\}$.

Case $x_0 = x_3$: The first term vanishes. Then

$$\mathrm{core}(x) = 900x_0 + 9x_1 + 90x_2 - 999x_4.$$

The maximum over $x_0, x_1, x_2 \in \{8, 9\}$ and $x_4 \in \{8, 9\}$ with $x_4 \leq x_3 = x_0$ is achieved at $x_0 = x_1 = x_2 = 9$, $x_4 = 8$: $900 \cdot 9 + 9 \cdot 9 + 90 \cdot 9 - 999 \cdot 8 = 8100 + 81 + 810 - 7992 = 999$. So $\mathrm{core}(x) \leq 999 < 10^3$.

Case $x_0 = x_3 + 1$: Then $x_0 = 9$ and $x_3 = x_4 = 8$ (forcing $x_4 = 8$ since $x_4 \leq x_3$). We get

$$\mathrm{core}(x) = 9000 + 900 \cdot 9 + 9x_1 + 90x_2 - 999 \cdot 8 = 9108 + 9x_1 + 90x_2$$

for $x_1, x_2 \in \{8, 9\}$ with $x_1 \geq x_2 \geq 8$. The minimum is $9108 + 72 + 720 = 9900$ and the maximum is $9108 + 81 + 810 = 9999$. So $\mathrm{core}(x) \in [9900, 9999]$, and in particular $\mathrm{core}(x) < 10^4$.

In both cases, $\mathrm{core}(x) < 10^4$. $\square$

We now complete the $d = 15$ closure argument.

Three cases for the second zero digit.

Case A: $x_4 = 0$. By sorted-descending, $x_5 = x_6 = \cdots = x_{14} = 0$. All five $\delta_k = 0$, so every block contributes $(0, 0)$ at decimals $[5, 6], [7, 8], \ldots, [13, 14]$. The block region produces 10 zero digits at decimals 5 through 14. Even after accounting for at most one carry from the core into decimal 5, decimals 6 through 14 remain zero — that is 9 zero digits, far more than the 2 required.

Case B1: $1 \le x_4 \le 7$. By Lemma C.7, $\sum_k \delta_k \le x_4 \le 7$. Since $\delta_k \ge 2$ for $k \in Z_{2+}$,

$$2Z_{2+} + Z_1 \le \sum_k \delta_k \le 7,$$

which gives

$$2Z_0 + Z_1 = 2M - Z_1 - 2Z_{2+} \ge 10 - 7 = 3.$$

So the block region contributes at least 3 zero digits.

Case B2: $x_4 \in \{8, 9\}$. By Lemma C.8, $\mathrm{core}(x) < 10^4$, so the padded core occupies only decimals 0 through 3, and decimal 4 is a zero digit in the core's contribution to $K^{(15)}(x)$. Combined with at least one zero digit from the block region (the $2Z_0 + Z_1 \ge 1$ bound), this gives at least 2 zero digits total.

In every case, $K^{(15)}(x)$ has at least 2 zero digits in its padded 15-digit representation, so its sorted-descending form ends in $(0, 0)$, which means $K^{(15)}(x) \in T_{15}$.

Combining: at every odd $d \ge 15$, $K^{(d)}(x)$ has at least 2 zero digits in its padded d-digit form, hence $K^{(d)}(x) \in T_d$. $\square$

Remark C.3 (at $d \le 13$, the bound fails). At $d = 13$ (odd), $M = 4$ and $2Z_0 + Z_1 \ge -1$ is vacuous. The algebraic argument does not apply. The finite-state enumeration of Proposition 5.2 handles $d \in \{7, 9, 11, 13\}$ directly, verifying reaching-time bounds of at most 8 iterations.

Corollary C.3 (computational confirmation). *Direct enumeration at $d = 15$ and $d = 17$ (odd ladder) over all admissible multisets (*1,307,404 *and* 3,124,450 *respectively) verifies one-step T_d closure: every admissible input reaches T_d in exactly one iteration.*

11.8 The even ladder

The even-ladder root at $d = 6$ uses the split-lifted rule with coefficient vector

$$c^{(6)} = (9900, 9, 90, -9000, 99000, -99999).$$

The split replaces the native $c_4 = -999$ with two coefficients at positions 4 and 5 summing to -999. For all higher even d, zero-sum pair extensions operate from $d = 6$.

Lemma C.4 (one-step T_d closure on the even ladder at $d \ge 18$). *Let $d \ge 18$ be even. For every admissible input $n \in A_d$ with sorted-descending form $(x_0, \ldots, x_{d-1})$,*

$$K^{(d)}_{60714}(n) \in T_d.$$

Equivalently, the padded d-digit representation of $K^{(d)}(n)$ has at least two zero digits.

Proof. The proof follows the same template as Lemma C.3 (odd ladder), with the $d_0 = 6$ split-root in place of the $d_0 = 5$ native root. We carry through the block-structure decomposition with the appropriate even-ladder adjustments.

Setup. At even $d \geq 8$, the coefficient vector is

$$c^{(d)} = \underbrace{(9900,\ 9,\ 90,\ -9000,\ 99000,\ -99999)}_{\text{split root, positions 0-5}} \parallel \underbrace{(+9 \cdot 10^7, -9 \cdot 10^7)}_{\text{pair 0, positions 6,7}} \parallel \cdots \parallel \underbrace{(+9 \cdot 10^{d-1}, -9 \cdot 10^{d-1})}_{\text{pair } M'-1\text{, positions } d-2, d-1}$$

where $M' = (d-6)/2$ is the number of appended pairs.

Block-structure decomposition. For $x = (x_0, \ldots, x_{d-1})$ sorted descending, define $\delta_k = x_{2k+6} - x_{2k+7}$ for $k = 0, 1, \ldots, M'-1$, so $\delta_k \in \{0, 1, \ldots, 9\}$. Then

$$K^{(d)}(x) = \mathrm{core}_{\mathrm{even}}(x) + \sum_{k=0}^{M'-1} 9 \cdot 10^{2k+7}\, \delta_k$$

where $\mathrm{core}_{\mathrm{even}}(x) = 9900x_0 + 9x_1 + 90x_2 - 9000x_3 + 99000x_4 - 99999x_5$ depends only on the first 6 positions, and is non-negative (Lemma C.9 below).

Decimal-block structure. Each block term $9 \cdot 10^{2k+7}\delta_k$ occupies decimal positions $[2k+7, 2k+8]$ as a two-digit value. Writing $9\delta_k = 10u_k + v_k$ for $u_k, v_k \in \{0, \ldots, 9\}$:

δ_k	$9\delta_k$	digit at $2k+7$ (v_k)	digit at $2k+8$ (u_k)
0	0	0	0
1	9	9	0
2	18	8	1
3	27	7	2
$\vdots$	$\vdots$	$\vdots$	$\vdots$
9	81	1	8

Block decimal positions are pairwise disjoint and disjoint from the core. Block k's contribution affects only decimal positions $2k+7$ and $2k+8$. Block $k+1$'s contribution affects positions $2(k+1)+7 = 2k+9$ and $2(k+1)+8 = 2k+10$. These do not overlap. The maximum block-k value is $81 \cdot 10^{2k+7}$ (when $\delta_k = 9$), which has digits at positions $2k+7$ and $2k+8$ but not at $2k+9$ — so blocks do not carry into one another. The core occupies decimal positions 0 through 5 (since $\mathrm{core}_{\mathrm{even}}(x) \leq 899{,}991 < 10^6$ by Lemma C.8 below). Decimal position 6 is occupied by neither core nor any block, so it is identically 0 in $K^{(d)}(x)$.

Lemma C.6 generalized to the even ladder. For sorted-descending x,

$$\sum_{k=0}^{M'-1} \delta_k = \sum_{k=0}^{M'-1} (x_{2k+6} - x_{2k+7}) \le x_6 - x_{d-1} \le 9.$$

The first inequality follows by telescoping: $\sum_k (x_{2k+6} - x_{2k+7}) = x_6 - x_{d-1} - \sum_{k=1}^{M'-1} (x_{2k+5} - x_{2k+6})$, and the inner gaps $x_{2k+5} - x_{2k+6}$ are non-negative by sorted-descending. The second is the trivial digit bound. This is the cliff-sum bound.

Block-zero count. Let $Z_0 = |\{k : \delta_k = 0\}|$, $Z_1 = |\{k : \delta_k = 1\}|$, and $Z_{2+} = |\{k : \delta_k \ge 2\}|$, so $Z_0 + Z_1 + Z_{2+} = M'$. From the table: - Each $\delta_k = 0$ block contributes 2 zero digits to $K^{(d)}(x)$ (at positions $2k+7$ and $2k+8$). - Each $\delta_k = 1$ block contributes 1 zero digit (at position $2k+8$, since $u_k = 0$). - Each $\delta_k \ge 2$ block contributes 0 zero digits at its own positions (both v_k and u_k are nonzero).

Total zero digits from blocks: $2Z_0 + Z_1$.

Plus position 6 is always zero (neither core nor any block touches it). So $K^{(d)}(x)$ has at least $2Z_0 + Z_1 + 1$ zero digits in its padded d-digit form.

From Lemma C.6, $\sum_k \delta_k \le 9$. Since $\delta_k \ge 2$ for each $k \in Z_{2+}$:

$$Z_1 + 2Z_{2+} \ \le\ Z_1 + \sum_{k \in Z_{2+}} \delta_k \ \le\ \sum_k \delta_k \ \le\ 9.$$

Substituting $Z_0 = M' - Z_1 - Z_{2+}$:

$$2Z_0 + Z_1 \ =\ 2M' - Z_1 - 2Z_{2+} \ \ge\ 2M' - 9.$$

The threshold. At even $d \ge 18$, $M' \ge 6$, hence $K^{(d)}(x)$ has at least $(2M'-9)+1 = 2M'-8 \ge 4$ zero digits in its padded d-digit form. Sorting $K^{(d)}(x)$'s digits in descending order places those zeros at the trailing positions; four trailing zeros suffices for T_d membership (which only requires two). □

Lemma C.8 (even-ladder core bound). *For every sorted-descending* $x = (x_0, \dots, x_5) \in \{0, \dots, 9\}^6$, $0 \le \mathrm{core}_{\mathrm{even}}(x) \le 899{,}991 < 10^6$.

Proof. Direct enumeration over the $\binom{15}{6} = 5{,}005$ sorted-descending 6-tuples confirms $\min \mathrm{core}_{\mathrm{even}} = 0$ (achieved at $(0,0,0,0,0,0)$, repdigit cases, and others) and $\max \mathrm{core}_{\mathrm{even}} = 899{,}991$ (achieved at $(9,9,9,9,9,0)$: $89{,}100 + 81 + 810 - 81{,}000 + 891{,}000 = 899{,}991$). Hence $\mathrm{core}_{\mathrm{even}} < 10^6$ uniformly. □

Lemma C.9 (even-ladder core non-negativity). *For every sorted-descending* $x \in \{0, \dots, 9\}^6$, $\mathrm{core}_{\mathrm{even}}(x) \ge 0$.

Proof. Direct enumeration over the 5,005 sorted-descending 6-tuples confirms $\mathrm{core}_{\mathrm{even}}(x) \ge 0$ in every case, with minimum 0. □

Corollary C.4 (computational confirmation). *Direct enumeration confirms one-step T_d closure on the even ladder at $d \ge 18$ (the threshold of Lemma C.4) and bounded reaching time at lower even d:*

- $d \in \{8, 10, 12, 14\}$*: max reaching time is* $6, 3, 2, 2$ *iterations respectively. Verified by Proposition 5.2's finite-state enumeration. (Lemma C.4's one-step bound does not apply at these d values; the bounded reaching time is what enters the induction in §5.7.)*

- $d \in \{16\}$*: every admissible multiset reaches* T_{16} *in exactly* 1 *step. Verified by Proposition 5.2's finite-state enumeration.*
- $d = 18$*: all* 4,686,725 *admissible multisets reach* T_{18} *in* ≤ 1 *step. Verified exhaustively via* `verify_even_ladder_closure.py 18`.
- $d = 20$*: all* 10,014,905 *admissible multisets reach* T_{20} *in* ≤ 1 *step. Verified exhaustively via* `verify_even_ladder_closure.py 20`.
- $d \in \{22, 24, 26\}$*: random sampling of* 200,000 *admissibles per* d *finds zero counterexamples. Verified by* `verify_even_ladder_closure.py D --sample 200000` *for* $D \in \{22, 24, 26\}$.

Remark C.4 (uniform reaching-time bound). Combining Lemma C.3 (odd ladder, $d \geq 15$) and Lemma C.4 (even ladder, $d \geq 18$) with the finite-state enumeration of Proposition 5.2 (covering odd $d \leq 13$ and even $d \leq 16$), every admissible input on either ladder at every $d \geq 5$ reaches T_d in a uniformly bounded number of iterations. Specifically: $T_d^* = 1$ for odd $d \geq 15$ and even $d \geq 18$; $T_d^* \leq 8$ for the remaining low-dimensional cases (Proposition 5.2). The induction in §5.7 closes on both ladders at every $d \geq 5$.

11.9 Admissible projection under Lemma 5.1

Lemma C.5 (admissible projection). *Let* $d \geq 7$*, let* $n \in A_d \cap T_d$ *with sorted-descending form* $(x_0, x_1, \ldots, x_{d-3}, 0, 0)$*, and let* m *be the integer with sorted-descending form* $(x_0, x_1, \ldots, x_{d-3})$ *at digit length* $d - 2$*. Then one of the following holds:*

1. $m \in A_{d-2}$ *(admissible at* $d-2$*): the inductive step of Theorem 5.2 applies directly, and the orbit of* n *reaches* 60714.
2. m *is a repdigit at* $d - 2$*:* n *belongs to a* ***residual escape class*** *and the orbit of* n *under* $K^{(d)}$ *reaches* 0 *rather than* 60714.
3. m *is a near-repdigit at* $d - 2$*:* $n \in A_d$ *(admissible at* d*), and after one iteration of* $K^{(d)}$ *the projection becomes admissible at* $d - 2$*, so the inductive step closes with a one-step delay; the orbit reaches* 60714.

Proof. We check the admissibility conditions at $d - 2$ in cases.

Case 1: m **is a repdigit at** $d - 2$ (i.e., $x_0 = x_1 = \cdots = x_{d-3} = v$ for some $v \in \{0, \ldots, 9\}$).

Sub-case 1a: $v = 0$. Then n's sorted-descending form is $(0, 0, \ldots, 0)$, i.e., $n = 0$. This is not in A_d, so this sub-case is excluded.

Sub-case 1b: $v \geq 1$. Then n's sorted-descending form is $(v, v, \ldots, v, 0, 0)$ with $d - 2$ copies of v and 2 zeros. This is admissible at d as long as $d - 2 \leq d - 2$ (trivially) and the zero count is $< d - 1$ (true for $d \geq 3$). So $n \in A_d$.

Applying $K^{(d)}$ to n: by Lemma 5.1, $K^{(d)}(n) = K^{(d-2)}(m)$ where $m = (v, v, \ldots, v)$ is a repdigit at $d - 2$. By Proposition 2.3, $K^{(d-2)}(\text{repdigit}) = 0$. So $K^{(d)}(n) = 0$, and the orbit reaches zero rather than 60714.

These inputs form the **residual escape class** for 60714's lifting at d. Their count per d equals the number of repdigit values $v \in \{1, \ldots, 9\}$, multiplied by (if we count integer inputs rather than multisets) the number of distinct permutations of the multiset, which is bounded by a small constant.

Case 2: m **is a near-repdigit at** $d - 2$ (one digit appears $d - 3$ times among $(x_0, \ldots, x_{d-3})$).

If the near-repeated digit is 0, then n has at least $d-3+2=d-1$ zero digits, making n itself near-repdigit at d (excluded from A_d).

If the near-repeated digit is some nonzero v, then n's sorted-descending form has $d-3$ copies of v, one other nonzero digit w, and two zeros. The digit count $(d-3,1,2)$ does not make n near-repdigit at d (requires $d-1$ or d of a kind), so $n \in A_d$.

For these inputs, the projection m at $d-2$ is a near-repdigit, so $m \notin A_{d-2}$ and the inductive hypothesis does not apply directly to m. However, the inductive step closes after at most two iterations of $K^{(d)}$ via the following structural fact.

Lemma C.10 (admissibility recovery for Case 2). *For every $d \geq 7$ and every Case 2 input $n \in A_d \cap T_d$ with near-repdigit projection m at $d-2$, the orbit $n, K^{(d)}(n), (K^{(d)})^2(n)$ contains a state $n^* \in T_d$ whose projection $m^* \in A_{d-2}$ is admissible at $d-2$. The number of iterations of $K^{(d)}$ required is at most 2.*

Proof. By Lemma 5.1, $K^{(d)}(n)$ is the integer $K^{(d-2)}(m)$, and $K^{(d)}(n) \in T_d$. The state $n_1 := \sigma_d(K^{(d)}(n))$ — i.e., the sorted-descending form at length d of the integer $K^{(d-2)}(m)$ — has a projection at $d-2$ given by its first $d-2$ entries. The proof of the lemma reduces to a case enumeration over the Case 2 inputs $n \in A_d \cap T_d$ — those whose sorted-descending form is $(v, v, \ldots, v, w, 0, 0)$ with $d-3$ copies of $v \neq 0$, one digit $w \neq v$, and two trailing zeros. (The $v=0$ row is excluded by admissibility at d: it would give $d-1$ zeros and one nonzero, a near-repdigit at d.) Of the 90 pairs (v, w) with $v \neq w$, exactly 81 produce admissible Case 2 inputs at d, partitioned as follows.

We separate near-repdigits at $d-2$ by the position of the unique unequal digit. A sorted-descending near-repdigit has the form $(v, v, \ldots, v, w)$ with $d-3$ copies of v and one digit $w \neq v$. Since the form is sorted descending, either $w > v$ (singleton at top, $m = (w, v, \ldots, v)$) or $w < v$ (singleton at bottom, $m = (v, v, \ldots, v, w)$).

The integer $K^{(d-2)}(m)$ has a closed form in each subcase, yielding a single decimal value whose sort at length d has explicit structure:

Case A: $w > v$ (36 inputs at d). The projection $m = (w, v, v, \ldots, v)$ at $d-2$ has the singleton at the top. $K^{(d-2)}(m)$ depends on w, v and the specific ladder; direct enumeration over all 36 Case A inputs (the 45 pairs with $w > v$ minus the 9 with $v = 0$, excluded by admissibility) at every $d-2 \in \{5, 6, \ldots, 14\}$ confirms that the resulting state n_1 has admissible-at-$(d-2)$ projection in every instance. The induction closes after one iteration of $K^{(d)}$.

Case B: $w < v$ (45 inputs at d). The projection $m = (v, v, \ldots, v, w)$ at $d-2$ has the singleton at the bottom. For $d-2 \geq 7$, the lifting structure (zero-sum appended pairs with $v > w$ pushed against the negative-coefficient slot) forces the closed form $K^{(d-2)}(m) = 9(v-w) \cdot 10^{d-3}$. (At $d-2 \in \{5, 6\}$ — i.e., $d \in \{7, 8\}$ — the closed form does not apply because the lifting at $d-2 = 5$ has the native -999 coefficient and at $d-2 = 6$ the split-lifted root is in play; these base cases are handled below.) Let $\Delta = v - w \in \{1, 2, \ldots, 9\}$.

The integer $9\Delta \cdot 10^{d-3}$, when sorted-descending at length d (note d, not $d-2$), produces a state whose first $d-2$ entries determine the projection m_1 at $d-2$. Direct computation:

- If $\Delta \in \{2, 3, \ldots, 9\}$ (36 Case B inputs total): $9\Delta \in \{18, 27, 36, 45, 54, 63, 72, 81\}$ has two nonzero digits, and $9\Delta \cdot 10^{d-3}$ at sort-desc-length-d has projection at $d-2$ that is an admissible multiset (two distinct nonzero digits, $d-4$ zeros — max count $d-4 < d-3$ for $d \geq 8$). The induction closes after one iteration of $K^{(d)}$.

- If $\Delta = 1$ (9 Case B inputs total): $9\Delta = 9$ has one nonzero digit, and $9 \cdot 10^{d-3}$ at sort-desc-length-d has projection at $d-2$ equal to $(9, 0, 0, \ldots, 0)$ — itself a near-repdigit at $d-2$. A second iteration of $K^{(d)}$ is required.

For these 9 second-iteration stragglers, applying $K^{(d)}$ again — which by Lemma 5.1 again projects to $K^{(d-2)}((9, 0, 0, \ldots, 0)) = 9 \cdot 9900 = 89{,}100$ on the odd ladder (the even ladder yields the analogous value). The integer 89,100, sorted-descending at length d, has projection $(9, 8, 1, 0, 0, \ldots, 0)$ at $d-2$ — admissible (three distinct nonzero digits, max count $d-5 < d-3$ for $d \geq 8$). The induction closes after two iterations of $K^{(d)}$.

Special-cases at the low end of the ladder. At $d-2 = 5$ (i.e., $d = 7$), the closed form for Case B does not apply, but direct enumeration shows all 81 admissible Case 2 inputs at $d = 7$ produce admissible-at-5 projections in one iteration; no second-iteration stragglers exist. At $d-2 = 6$ (i.e., $d = 8$), the closed form for Case B again does not apply directly, but direct enumeration confirms the same general pattern: 72 recover in 1 iteration (36 Case A + 36 Case B with $\Delta \geq 2$), 9 recover in 2 iterations (9 Case B with $\Delta = 1$).

Total. For every $d \geq 8$: of the 81 admissible Case 2 inputs at d, 72 recover an admissible-at-$(d-2)$ projection after 1 iteration of $K^{(d)}$ and the remaining 9 recover after 2 iterations. At $d = 7$, all 81 recover after 1 iteration. The induction in §5.7 closes in either case. □

Note. The script `verify_case2_recovery.py` provides per-d verification of the 72/9 split (or 81/0 at $d = 7$) and confirms all reach 60714, with the maximum total reaching time bounded uniformly: 27 steps on the odd ladder, 15 on the even ladder, independent of d (verified through $d = 30$).

Empirical confirmation across the verified range. Exhaustive enumeration at every $d \in \{7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$ confirms: all 81 Case 2 multisets per d reach 60714, with maximum reaching times $29, 15, 27, 15, 27, 15, 27, 15, 27, 15, 27, 15, 27, 15$ steps respectively (verified by `verify_case2_recovery.py`).

Case 3: $m \in A_{d-2}$ (admissible at $d-2$, neither repdigit nor near-repdigit). The inductive hypothesis applies: by Theorem 5.2 at $d-2$, the orbit of m under $K^{(d-2)}$ reaches 60714 in finitely many steps. By Lemma 5.1, the orbit of n under $K^{(d)}$ tracks this orbit and reaches $60714_{(d)}$ (i.e., 60714 padded to d digits).

Summary. For $n \in A_d \cap T_d$ with $m \in A_{d-2}$ (Case 3), the induction closes directly. For n in the residual escape class (Case 1b only — repdigit projection), the orbit reaches 0 rather than 60714. For n with near-repdigit projection (Case 2), the induction closes with a one-step delay: $K^{(d)}(n)$ has admissible projection at $d-2$, and the inductive hypothesis takes over from there. The main theorem's statement — universal at every $d \geq 5$ — holds in the strict sense at $d = 5$ and $d = 6$ (where the escape class is empty) and holds *modulo the characterized residual escape class* (Case 1b only) at $d \geq 7$. The escape class is explicitly documented at each d in Appendices D and E. □

Remark C.6. The statement of Theorem 5.2 (§5.2) makes the dependence on the escape class E_d precise: universal convergence holds on $A_d \setminus E_d$ at every $d \geq 5$, with $E_d = \emptyset$ at $d = 5, 6$ and E_d non-empty at $d \geq 7$. The size of the step-1 component $E_d^{(1)}$ admits the closed form (Lemma 5.5)

$$|E_d^{(1)}| \;=\; \binom{10+k-1}{k} - 10, \qquad k = 1 + \lfloor (d-d_0)/2 \rfloor.$$

Numerically, $|E_d^{(1)}|$ is $45, 45, 210, 210, 705, 705$ at $d = 7, 8, 9, 10, 11, 12$, growing polynomially in d. The full escape class E_d is the backward orbit of the block-aligned set under $K^{(d)}_{60714}$; its asymptotic size is bounded by direct enumeration at each d but no closed form is currently proven. Empirically, every orbit in E_d at $d \leq 11$ collapses to 0 within 4 iterations.

12 Appendix D. Universal Full-Variable Fixed Points at $d = 7$

This appendix documents the classification of universal full-variable fixed points at $d = 7$ and the cross-dimensional outcomes for the two-zero $d = 6$ stratum tested at $d = 7$. Two complementary enumerations underlie the data:

- **Run B (exhaustive classification at $d = 7$).** Sound exhaustive enumeration over all $9{,}344{,}160 = 7! \cdot D_7$ full-variable rules at $d = 7$, identifying every universal full-variable fixed point. Numba-accelerated, approximately 45 seconds wall-time on commodity hardware.
- **Run C (cross-dimensional outcome verification).** For each of the 53 two-zero $d = 6$ universal fps, exhaustive enumeration of all rank-7 rules fixing the fp with exact basin computation. Approximately one hour wall-time.

A held-out validation set of 12 additional $d = 6$ fps (drawn from 0-zero, 1-zero, 2-zero, and 3-zero strata) is used in §D.2 to validate the Type A LOCK characterization beyond the two-zero stratum. The complete computational pipeline, including the audit package `d7_audit/`, is provided as supplementary material (§1.7.1).

12.1 Universal full-variable fps at $d = 7$

Run B identifies 18,004 **universal full-variable fixed points at $d = 7$**, attained by 49,292 universal rules. Zero-stratification:

Zero count	Universal fixed points
0	7,541
1	8,125
2	1,996
3	316
4	19
5	7

The 21 strict-universal $d = 7$ fps catalogued in earlier drafts of this Appendix (under "Case 1" and "Case 2") all appear in the Run B catalog with rule-count metadata matching exactly. The earlier draft also listed $F = 1{,}406{,}070$ as a "NEAR-edge" case; this fp does not appear in the Run B catalog of strict-universal fps because its best basin at $d = 7$ is below the strict-universal threshold (basin $\approx 0.9999 < 1$), confirming the earlier NEAR-edge characterization.

12.2 Cross-dimensional outcomes: the two-zero $d = 6$ stratum

Of the 506 universal full-variable fixed points at $d = 6$, exactly 53 have multiset structure with two trailing zeros at $d = 6$ (equivalently, three trailing zeros when padded to $d = 7$). Run C performs exhaustive outcome verification for these 53 fps at $d = 7$, classifying each as TRANS (best basin ≥ 0.99), NEAR (basin in $[0.95, 0.99)$), LOCK (basin < 0.95), or algebraic obstruction (no fixing rule exists).

Result. 43 TRANS, 2 NEAR, 8 LOCK. Of the 8 LOCK fps, 4 are Type A LOCK and 4 are Type B LOCK in the sense of §6.6. The 43 TRANS comprise 42 strict-universal (basin $= 1$) plus $F = 60714$ (basin $= 0.992857$, NEAR-strict by the threshold of §1.4).

Strict-transcendence rate for the two-zero $d = 6$ stratum at $d = 7$: $43/53 \approx 81.1\%$. Including NEAR cases the rate is $45/53 \approx 84.9\%$. Earlier estimates based on the smaller 21-fp strict-universal subset (table D.7 of the prior draft) suggested cross-dimensional rates near 96%; the corrected figure for the full 53-fp stratum is lower, though still high enough to justify the qualitative claim that cross-dimensional transcendence is "common at higher d_F."

The full verified outcomes table:

F	Outcome	Best basin	Type	Valid rules	Rules ≥ 0.99
4,545	TRANS	1.000000	–	1,920	12
7,191	NEAR	0.984568	–	432	0
8,919	TRANS	1.000000	–	768	14
8,991	TRANS	1.000000	–	1,632	68
9,189	TRANS	1.000000	–	1,344	8
9,225	LOCK	0.884215	B	216	0
10,899	TRANS	1.000000	–	1,632	16
17,019	LOCK	0.574250	A	192	0
37,017	LOCK	0.860582	A	192	0
44,505	TRANS	1.000000	–	1,920	4
52,605	LOCK	0.563404	B	288	0
54,450	TRANS	1.000000	–	1,920	10
60,714	TRANS	0.992857	–	36	4
67,023	LOCK	0.868783	B	36	0
80,919	TRANS	1.000000	–	1,584	30
80,991	TRANS	1.000000	–	768	26
81,099	TRANS	1.000000	–	768	8
89,019	TRANS	1.000000	–	2,208	20
90,819	TRANS	1.000000	–	2,112	4
90,918	TRANS	1.000000	–	1,344	4
91,809	TRANS	1.000000	–	1,440	6
98,091	TRANS	1.000000	–	1,440	16
100,899	TRANS	1.000000	–	2,400	46
108,099	TRANS	1.000000	–	1,440	18
109,809	TRANS	1.000000	–	1,296	4
146,070	TRANS	1.000000	–	108	4
170,460	LOCK	0.922928	B	36	0
180,909	TRANS	1.000000	–	1,440	6
189,009	TRANS	1.000000	–	2,784	8
190,089	TRANS	1.000000	–	2,208	22
190,809	TRANS	1.000000	–	768	30
445,005	TRANS	1.000000	–	1,920	10
445,050	TRANS	1.000000	–	1,920	12
504,450	TRANS	1.000000	–	1,920	14
544,500	TRANS	1.000000	–	1,920	12

F	Outcome	Best basin	Type	Valid rules	Rules ≥ 0.99
607,140	NEAR	0.985450	–	36	0
700,191	TRANS	1.000000	–	432	2
701,901	LOCK	0.816755	A	192	0
707,130	LOCK	0.916314	A	384	0
719,100	TRANS	1.000000	–	432	2
809,091	TRANS	1.000000	–	2,880	24
809,109	TRANS	1.000000	–	2,208	16
810,909	TRANS	1.000000	–	2,976	10
890,091	TRANS	1.000000	–	768	14
900,891	TRANS	1.000000	–	768	24
901,890	TRANS	1.000000	–	1,440	6
908,091	TRANS	1.000000	–	2,208	28
908,109	TRANS	1.000000	–	2,304	18
908,190	TRANS	1.000000	–	2,112	14
908,901	TRANS	1.000000	–	1,536	36
909,081	TRANS	1.000000	–	2,976	34
910,089	TRANS	1.000000	–	768	44
910,809	TRANS	1.000000	–	2,208	14

The "Type" column is – for TRANS/NEAR fps; A or B for LOCK fps per the §6.6 classification. The "Rules ≥ 0.99" column counts $d = 7$ fixing rules whose basin meets the strict-universal threshold of §1.4.

12.3 Held-out validation of the Type A LOCK characterization

To validate the Type A characterization (Definition 6.3) beyond the two-zero stratum, we ran the verifier on 12 additional $d = 6$ universal fps drawn from 0-zero, 1-zero, 2-zero, and 3-zero strata (3 each):

F	Stratum at $d = 6$	Outcome at $d = 7$	Type	zsp rules / total rules at $d = 7$
112,743	0-zero	LOCK	A	0/32
284,679	0-zero	LOCK	B	6/16
477,936	0-zero	(algebraic obstruction)	–	0/0
11,736	1-zero	LOCK	A	0/16
175,086	1-zero	LOCK	B	8/12
386,064	1-zero	LOCK	A	0/16
4,545	2-zero	TRANS	–	1,344/1,920
89,019	2-zero	TRANS	–	1,248/2,208
544,500	2-zero	TRANS	–	1,344/1,920
252	3-zero	TRANS	–	1,152/4,224
20,025	3-zero	TRANS	–	1,152/4,224
200,025	3-zero	TRANS	–	1,152/4,224

The Type A predictor was tested blind on this held-out set: 3 of 12 fps (112,743, 11,736, 386,064) were predicted Type A LOCK based solely on their absence of zsp rules at $d = 7$. All three predictions matched the verified LOCK outcomes. No false positives occurred (no Type A prediction fired on any of the 6 TRANS records or 2 Type B LOCK records).

The held-out result confirms that Type A LOCK is not specific to the two-zero stratum; it is a general structural mechanism applicable across multiple zero-count strata at $d = 6 \to d = 7$. Combined with Run C, the Type A predictor has 7 correct predictions out of 7 across 65 total verified records, with zero false positives.

12.4 Cycle signatures at $d = 7$

The cycle signature of a permutation pair (π, σ) records the cycle decomposition of $\pi \cdot \sigma^{-1}$. Among the 53 Run C fps, four signatures appear among fixing rules: $(2, 2, 3)$, $(2, 5)$, $(3, 4)$, and $(7,)$. The signature $(3, 4)$ — the natural structural analog at $d_F = 7$ of $(3, 3)$ at $d_F = 6$ — does *not* uniformly produce LOCK rules: 9 of the 53 fps include $(3, 4)$ in their signature set and TRANS, while 5 include $(3, 4)$ and LOCK. The factorization mechanism that locks $(3, 3)$ at $d_F = 6$ (Theorem 6.1 part 4) does not generalize directly to $(3, 4)$ at $d_F = 7$.

A refined cycle-signature pattern does hold (§6.7, observation 4): fps whose fixing-rule space is restricted to only the long-cycle signatures $\{(3, 4), (7,)\}$ uniformly LOCK at $d = 7$ (4 of 4 in Run C). All four cases are also Type A LOCK by Observation 6.3, which suggests the cycle-signature restriction is a *consequence* of the more fundamental zsp absence rather than an independent classifier.

12.5 Cross-dimensional behavior

Transition	Universal fps	Strict TRANS	Rate
$d = 4 \to d = 5$	4 at $d = 4$	0 at $d = 5$	0%
$d = 5 \to d = 6$	33 at $d = 5$	1 at $d = 6$ (60,714)	3%
$d = 6 \to d = 7$ (two-zero stratum)	53	43	81%
$d = 6 \to d = 7$ (held-out 12-fp set)	11 (excluding alg. obstruction)	6	55%

Observation D.1 (revised, empirical). *Cross-dimensional transcendence from d_F to $d_F + 1$ rises sharply with the zero-digit count of F's padded multiset. Within the two-zero $d = 6$ stratum tested at $d = 7$, the strict-transcendence rate is $43/53 \approx 81\%$. Within the broader held-out set spanning 0-zero through 3-zero strata, the rate falls to $6/11 \approx 55\%$, with 0-zero and 1-zero fps disproportionately LOCK or algebraically obstructed. The trend is consistent with Conjecture 7.2: dimension-transcendence is enabled by absorbing capacity, which grows with both d_F and zero-digit count.*

Note on the 4-zero stratum at $d = 6$. No universal full-variable fixed points exist at $d = 6$ with 4 or more zero digits in their padded sorted-descending form, by exhaustive enumeration. The trivial fixed point $F = 0$ (with 6 zero digits) is excluded throughout this paper by the convention of §2.

12.6 Reproducibility

The complete data underlying this appendix is reproducible from the supplementary `d7_audit/` package (§1.7.1):

- `classify_d7_full.py` reproduces Run B (the 18,004-fp classification at $d = 7$). Wall-time: ≈ 45 s.
- `d7_outcome_verifier.py` reproduces Run C (cross-dimensional outcomes for the two-zero $d = 6$ stratum). Wall-time: ≈ 1 hour.
- `audit_60714_d7.py` and `audit_60714_d8.py` reproduce the empirical confirmation of Lemma 5.5 at $d = 7$ and $d = 8$ (cf. §5 Remark). Wall-time: ≈ 3 s each.

Per-fp data is recorded in `d7_classification.json` (Run B output) and `d7_verified_outcomes.json` (Run C output), both included as supplementary files. The earlier `dF7_summary_complete.json` is retained for reference; its 21-fp strict-universal listing is a strict subset of Run B's 18,004-fp catalog.

End of Appendix D.

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13 Appendix E. Audit of 6174 at $d = 8, 9$

This appendix documents the exhaustive audit of coefficient-preserving liftings of 6174's native rule at digit lengths $d = 8$ and $d = 9$, supporting Theorem 6.1 in §6.

13.1 Setup

The native rule of 6174 at $d_F = 4$ is the classical Kaprekar rule K_0: $\pi = (3, 2, 1, 0)$, $\sigma = (0, 1, 2, 3)$, coefficient vector $c^{(4)} = (999, 90, -90, -999)$. The sorted-descending form of 6174 at $d = 4$ is $(7, 6, 4, 1)$ with no zero digits, so every native coefficient lands on a nonzero digit: $\mathrm{sv}_F(K_0) = \mathrm{sv}(K_0) = 4$.

When we pad 6174 to $d = 8$, the sorted-descending form becomes $(7, 6, 4, 1, 0, 0, 0, 0)$. The four zero positions (indices $4, 5, 6, 7$) become the "absorbing" positions for coefficient-preserving lifting: new coefficients at these positions multiply zero digits and do not affect $K(F) = F$.

Coefficient-preserving lifting at $d = 8$. A rule K at $d = 8$ with coefficient vector $c = (c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7)$ is a coefficient-preserving lifting of K_0 if $c_i = c_i^{(4)}$ for $i \in \{0, 1, 2, 3\}$ — that is, if the first four coefficients are exactly $(999, 90, -90, -999)$. By Proposition 5.1, any such rule satisfies $K(6174_{(8)}) = 6174$.

Given the native constraint $c_i = c_i^{(4)}$ at positions $0, 1, 2, 3$, the permutations π and σ are forced at those positions: $\pi_0 = 3, \sigma_0 = 0$, $\pi_1 = 2, \sigma_1 = 1$, $\pi_2 = 1, \sigma_2 = 2$, $\pi_3 = 0, \sigma_3 = 3$. The remaining positions $4, 5, 6, 7$ are freely assignable, subject to:

- $(\pi_4, \pi_5, \pi_6, \pi_7)$ is a permutation of $\{4, 5, 6, 7\}$: $4! = 24$ choices.
- $(\sigma_4, \sigma_5, \sigma_6, \sigma_7)$ is a permutation of $\{4, 5, 6, 7\}$: 24 choices.
- $\pi_i \neq \sigma_i$ for $i \in \{4, 5, 6, 7\}$ (derangement at the tail): $D_4 = 9$ choices given π.

Total liftings at $d = 8$: $24 \cdot 9 = 216$.

Analogous setup at $d = 9$. The sorted-descending form of 6174 at $d = 9$ is $(7, 6, 4, 1, 0, 0, 0, 0, 0)$ with five zero positions. Coefficient-preserving liftings preserve $c^{(4)}$ at positions $0, 1, 2, 3$ and freely assign π, σ at positions $4, 5, 6, 7, 8$. Total count:

$$5! \cdot D_5 = 120 \cdot 44 = 5{,}280.$$

13.2 Audit results

13.3 At $d = 8$

Exhaustive audit of all 216 coefficient-preserving liftings at $d = 8$, each tested against all 24,210 admissible multisets with iteration budget 50:

Status	Count
Strict-universal (basin = 1.0)	0
Near-universal (basin ≥ 0.99)	15
Sub-near ($0 \leq$ basin < 0.99)	201
Best basin	0.998141

Runtime: approximately 150 seconds on commodity hardware.

Best lifting (representative). Among the two liftings achieving the maximum basin 0.998141 (a sign-flip pair), one has permutations

$$\pi = (3, 2, 1, 0, 5, 7, 6, 4), \qquad \sigma = (0, 1, 2, 3, 6, 4, 5, 7),$$

with coefficient vector

$$c = (999,\ 90,\ -90,\ -999,\ -900{,}000,\ 9{,}990{,}000,\ 900{,}000,\ -9{,}990{,}000).$$

Verification: $K(6174_{(8)}) = |999{\cdot}7{+}90{\cdot}6{-}90{\cdot}4{-}999{\cdot}1{+}0{+}0{+}0{+}0| = |6993{+}540{-}360{-}999| = 6174$. ✓

13.4 Escape characterization at $d = 8$

For the best lifting, exactly 45 admissible multisets fail to reach 6174. All 45 have the form of **two four-of-a-kind groups**: sorted-descending (X, X, X, X, Y, Y, Y, Y) for some pair $X > Y \geq 0$.

There are exactly $\binom{10}{2} = 45$ such digit pairs (X, Y) with $X > Y \geq 0$, accounting for all 45 escape multisets.

Complete list of escape multisets at $d = 8$ (listed as sorted-descending tuples):

$(1, 1, 1, 1, 0, 0, 0, 0)$, $(2, 2, 2, 2, 0, 0, 0, 0)$, $\ldots$, $(9, 9, 9, 9, 0, 0, 0, 0)$ — the 9 multisets with $Y = 0$.

$(2, 2, 2, 2, 1, 1, 1, 1)$, $(3, 3, 3, 3, 1, 1, 1, 1)$, $\ldots$, $(9, 9, 9, 9, 1, 1, 1, 1)$ — the 8 multisets with $Y = 1$.

$\vdots$

$(9,9,9,9,8,8,8,8)$ — the 1 multiset with $Y = 8$.

Total: $9+8+7+6+5+4+3+2+1 = 45$ **multisets.**

Why these escape. Each escape multiset (X,X,X,X,Y,Y,Y,Y) has only two distinct digit values, with exactly four copies of each. Applied to the best lifting:

$$\begin{aligned}
K(X^4Y^4) &= |c_0 \cdot X + c_1 \cdot X + c_2 \cdot X + c_3 \cdot X + c_4 \cdot Y + c_5 \cdot Y + c_6 \cdot Y + c_7 \cdot Y| \\
&= |(c_0 + c_1 + c_2 + c_3) \cdot X + (c_4 + c_5 + c_6 + c_7) \cdot Y| \\
&= |(999 + 90 - 90 - 999) \cdot X + (-900{,}000 + 9{,}990{,}000 + 900{,}000 - 9{,}990{,}000) \cdot Y| \\
&= |0 \cdot X + 0 \cdot Y| = 0.
\end{aligned}$$

The first four coefficients sum to 0 (since K_0 has sum-zero structure), and the last four coefficients sum to 0 (by the coefficient-preserving lifting construction). When the input has only two distinct values, the rule's action collapses algebraically to zero.

This is not a feature of the specific best lifting — it is a structural consequence of sum-zero constraints on coefficient-preserving liftings. Every coefficient-preserving lifting at $d = 8$ maps every four-of-a-kind-paired multiset to 0.

13.5 At $d = 9$

Exhaustive audit of all 5,280 coefficient-preserving liftings at $d = 9$, each tested against all 48,520 admissible multisets with iteration budget 60:

Status	Count
Strict-universal (basin = 1.0)	0
Near-universal (basin $\geq$ 0.99)	302
Best basin	0.999073

Runtime: approximately 60 minutes on commodity hardware.

13.6 Escape characterization at $d = 9$

For the best lifting at $d = 9$, the 45 escape multisets have the same structural form as at $d = 8$, extended with trailing zeros. Specifically: sorted-descending $(X,X,X,X,Y,Y,Y,Y,0)$ for $X > Y \geq 0$ (when $Y \geq 1$) or $(X,X,X,X,0,0,0,0,0)$ (when $Y = 0$). There are again $\binom{10}{2} = 45$ such multisets.

The escape count is independent of d. The 45-multiset structural form is preserved across $d = 8, 9$ and is expected to continue at $d \geq 10$: at higher d, the escape multisets are obtained by appending trailing zeros to the $d = 8$ escape forms.

13.7 Asymptotic behavior

Basin growth. The escape count at $d = 8, 9$ is fixed at 45. The admissible multiset count grows with d:

d	# admissible multisets	# escapes	basin (best)
8	24,210	45	0.998141
9	48,520	45	0.999073
10	expected $\sim 92{,}000$	expected 45	expected ~ 0.9995
$\cdots$	$\cdots$	$\cdots$	$\cdots$
$d \to \infty$	$\to \infty$	45	$\to 1$

Asymptotic claim. The best-lifting basin for 6174 approaches 1 as $d \to \infty$. Formally, if the 45-escape structural form persists at all $d \geq 8$, then

$$\text{best basin at } d = 1 - \frac{45}{|A_d|} = 1 - O(10^{-d/9}).$$

The constant-count escape class means the ratio $45/|A_d|$ decays as $|A_d|$ grows, which for admissible multisets at digit length d grows combinatorially.

Strict-universal at any $d \geq 8$? Audit at $d = 8, 9$ finds 0 strict-universal liftings. Whether any $d \geq 10$ admits a strict-universal coefficient-preserving lifting of 6174 remains open. The structural argument of §E.2.2 suggests the answer is no: the four-of-a-kind-paired escape class is algebraically forced by sum-zero on the native and appended coefficients, and appears in every coefficient-preserving lifting.

13.8 Conjectural extension

Conjecture E.1. *For every $d \geq 8$, the 45-multiset escape class persists at 6174's coefficient-preserving liftings, and no coefficient-preserving lifting is strict-universal. The best basin asymptotes to 1 monotonically.*

Resolving Conjecture E.1 requires a structural argument at general d beyond the direct audit at $d = 8, 9$. The algebraic proof of the 45-multiset escape class's persistence (that it is forced by sum-zero constraints regardless of which coefficient-preserving lifting is chosen) is outlined in §E.2.2 and could likely be extended to all d, but we do not pursue this here.

End of Appendix E.

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14 Appendix F. Strict-d Anchors and the $\{7, 6, 4, 1\}$-Thread Tower

Status: Preliminary working notes added April 29, 2026 (revised April 30, 2026 to correct §F.5 — see "Note (added in revision)" below). The classifications and counts in §F.3 and §F.4 are computational findings established by two independent enumerations: the original `universality_scan_v3.py` run and a fresh from-scratch verifier (`verifier_v2.py`, fp-first traversal with direct (A, B) image-pair enumeration) that reproduces every count exactly. The strict-d criterion of §F.2 is well-defined and exactly captures the distinction we use; whether it is the right central object for a future revision of the main paper, and whether the reported $d = 7$ counts extend to higher d, are explicitly left open in §F.6.

14.1 Purpose

The main paper establishes that 60714 is a universal full-variable fixed point at every digit length $d \geq 5$ under the coefficient-preserving lifting family of §5 (Theorem 3). That result is unchanged.

The notion of universality used throughout the main paper admits absorption to 0: a rule K is universal for fixed point F at digit length d iff every input in A_d (the admissible inputs, excluding repdigits and near-repdigits) iterates under K either to F or to 0, with no input entering a non-$\{F, 0\}$ cycle. Inputs that absorb to 0 are the structurally characterized escape class of §5 (Lemma 5.1, Lemma 5.2). The present appendix makes this convention explicit and uses the abbreviation **F-or-0 universality** when the distinction matters.

This appendix introduces a strengthened criterion — *strict at d* — that distinguishes rules whose dynamics genuinely involve all d coefficient positions on F from rules that are zero-sum-pair extensions of lower-dimensional rules. Under the strict criterion, the cross-dimensional structure of the $\{7, 6, 4, 1\}$-thread is substantially richer than reported in the main paper, and the classical Kaprekar constants 6174 and 1746 acquire strict-anchor status at $d = 7$ — a finding consistent with, but sharper than, the cross-dimensional pattern of 6174 described in §6.

14.2 The strict-d criterion

Let $K = K_{\pi,\sigma}$ be a universal full-variable rule for F at digit length d, with coefficient vector $c = (c_0, c_1, \ldots, c_{d-1})$, and let $(x_0, x_1, \ldots, x_{d-1})$ denote F's sorted-descending form padded to d positions.

Definition F.1 *(trivial zero-sum pair on F).* A pair of indices (i, j) with $i < j$ is a *trivial zero-sum pair on F* if $c_i + c_j = 0$ and $x_i = x_j$. Such a pair contributes $c_i x_i + c_j x_j = c_i(x_i - x_j) = 0$ to $K(F)$, so the fixed-point equation $K(F) = F$ is unchanged if the pair is removed: the rule's action on F itself is reproduced by a $(d-2)$-coefficient sub-rule.

Definition F.2 *(strict at d).* A universal full-variable rule K for F at digit length d is *strict at d* if its coefficient vector has no trivial zero-sum pair on F in the sense of Definition F.1.

Definition F.3 *(strict-d anchor).* A fixed point F is a *strict-d anchor* if F is a universal full-variable fixed point at digit length d and admits at least one strict-at-d universal rule.

The main paper's results are stated in terms of universal fps; that is, fps admitting any universal full-variable rule. Strict-d anchors are the subset of universal fps for which the universality is not achieved purely by zero-sum-pair extension on equal-digit positions of F.

14.3 The {7,6,4,1}-thread under strict-d

We applied the strict-d criterion exhaustively at digit lengths $d = 5, 6, 7$ to all universal full-variable fps in the relevant zero-padded extensions of $\{7, 6, 4, 1\}$.

Observation F.1 *(strict-d anchors of the $\{7, 6, 4, 1\}$-thread).* The strict-d anchors at digit lengths $d \leq 7$ are:

d	Multiset	Strict-d anchors	Count
4	$\{7, 6, 4, 1\}$	1746, 6174	2
5	$\{7, 6, 4, 1, 0\}$	60417, 60714	2
6	$\{7, 6, 4, 1, 0, 0\}$	60714, 146070, 170460, 607140	4

d	Multiset	Strict-d anchors	Count
7	$\{7,6,4,1,0,0,0\}$	(see below)	11

The full $d = 7$ list is:

> 1746, 6174, 17460, 61740, 146070, 174006, 174600, 1,400,706, 1,460,700, 1,746,000, 6,174,000.

The $d = 6$ classification reproduces the four-fp set highlighted in §6.2.3 of the main paper. The $d = 7$ classification is new in this appendix.

The verification is computational. For each $d \in \{5, 6, 7\}$ we enumerated all sv$=d$ rules fixing each integer in the multiset, tested each fixer for universality (under the F-or-0 convention), and tested each universal rule for the strict-d criterion of Definition F.2. The classification was carried out by two independently developed scanners: `universality_scan_v3.py` (rule-first traversal) and `verifier_v2.py` (fp-first traversal, fresh from-scratch implementation). Both scanners produce the same lists at $d = 5, 6, 7$, the same rule-counts per fp, and the same actual rule sets where checked. Source code and JSON outputs are provided as supplementary material.

14.4 The non-monotone strict-anchor pattern

Observation F.2 *(non-monotone strict-anchor pattern of* 6174 *and* 1746*).* The classical Kaprekar fixed points 6174 and 1746 are strict-d anchors at $d = 4$ and $d = 7$, but not at $d = 5$ or $d = 6$. Specifically:

- *At $d = 5$:* neither 6174 nor 1746 admits any sv$=5$ rule satisfying $K(F) = F$ (algebraic obstruction). This matches §1.4 of the main paper.
- *At $d = 6$:* both 6174 and 1746 admit universal sv$=6$ rules under the F-or-0 convention, but every such rule is a zero-sum-pair extension of the classical $d = 4$ rule on the two zero positions. Under Definition F.2 these rules are non-strict.
- *At $d = 7$:* 6174 admits 6 strict-$d = 7$ universal rules and 1746 admits 12. The rules engage all seven coefficient positions in non-trivial pairings, with no pair summing to zero on equal digits of F.

This pattern was previewed in §6 of the main paper as the *cross-dimensional fingerprint* of 6174 (universal at $d = 4$ and near-universal at $d \geq 7$, obstructed at $d = 5, 6$), but stated in terms of universality rather than strict anchorhood. The strict-d criterion sharpens the observation: 6174's reappearance at $d = 7$ is not a quirk of a particular lifting family but a structural feature — 6174 is a strict-d anchor at $d = 7$ in the same sense it is at $d = 4$.

Observation F.3 *(strict-anchor status of* 60714 *and* 60417 *at $d = 7$).* Both 60714 and 60417 admit universal sv$=7$ rules at $d = 7$ under the F-or-0 convention, but every such rule contains a trivial zero-sum pair on F. Under Definition F.2, 60714 and 60417 are not strict-d anchors at $d = 7$.

For 60714, the canonical lifting of §5 of the main paper has coefficient vector

$$c = (9900,\ 9,\ 90,\ -9000,\ -999,\ 900000,\ -900000).$$

The pair $(900000, -900000)$ at indices $5, 6$ lands on the two trailing zero positions of the sorted-descending form sort_desc(60714) $= (7, 6, 4, 1, 0, 0, 0)$. Both positions hold digit 0, so the pair contributes zero on F, and the fixed-point equation is reproduced by the five-coefficient sub-rule at indices $\{0, 1, 2, 3, 4\}$ — which is precisely the $d = 5$ rule.

This observation is consistent with the main paper's Theorem 3: 60714 is a universal full-variable fp at every $d \geq 5$. The strengthening is that for $d \geq 7$, the lifting is non-strict by Definition F.2; the genuinely d-dimensional dynamics of 60714 live at $d = 5$ and $d = 6$.

14.5 Pure-duplication extensions

The main paper considers only zero-padded extensions of multisets across d. We tested the alternative — *pure-duplication extension*, where the digits added at $d > 4$ are drawn from the classical core itself (e.g., $\{7, 7, 6, 4, 1\}$ at $d = 5$).

Observation F.4 *(pure-duplication extensions at $d \in \{5, 6, 7\}$).* Let $M = \{7, 6, 4, 1\}$ be the $d = 4$ classical multiset. For $d \in \{5, 6\}$, every multiset of the form $M \cup S$ with S drawn from M has digit sum $18 + |S| \cdot \bar{m}$ where $\bar{m}$ ranges over averages of multisets of M. No such sum is divisible by 9 for $|S| \in \{1, 2\}$; consequently no universal fp exists in these multisets at $d = 5, 6$. For $d = 7$, four pure-duplication multisets are admissible by digit sum — namely $\{7, 7, 7, 6, 4, 4, 1\}$, $\{7, 7, 6, 4, 1, 1, 1\}$, $\{7, 6, 6, 6, 6, 4, 1\}$, $\{7, 6, 4, 4, 4, 1, 1\}$ — and exhaustive enumeration over all sv≥ 2 rules at $d = 7$ produces zero rule-fp pairs satisfying $K(F) = F$ in any of the four. The same result holds for $\{8, 5, 3, 2\}$ at $d = 5, 6, 7$.

Note (added in revision). A subsequent investigation extended this inquiry to $d = 8$. At $d = 8$ in the $\{7, 6, 4, 1\}$-thread, three of the four admissible pure-duplication multisets — $\{7, 7, 7, 7, 6, 6, 4, 1\}$, $\{7, 6, 6, 4, 4, 4, 4, 1\}$, $\{7, 6, 6, 4, 1, 1, 1, 1\}$ — remain empty of universal sv$= 8$ fps, but the *symmetric*-duplication multiset $\{7, 7, 6, 6, 4, 4, 1, 1\}$ (each core digit doubled exactly twice) admits universal full-variable rules. Pure-duplication extensions are therefore not uniformly empty across all $d > 4$, and zero-padding is not the unique viable extension mechanism. The pattern by which symmetric duplications break the empty-pure-duplication property at $d = 8$ — and whether this extends to higher even d or to the $\{8, 5, 3, 2\}$-thread — is open.

The empirical pattern at $d \in \{5, 6, 7\}$ remains: zero-padding is the viable extension mechanism for the strict-d tower at those digit lengths. The qualifier "at any $d > 4$" stated in earlier versions of this appendix is withdrawn; the correct scope is $d \in \{5, 6, 7\}$.

14.6 Open questions

This appendix reports observations; it does not settle them.

1. *Is the strict-d criterion of Definition F.2 the right structural object?* The criterion captures the zero-sum-pair-on-equal-digits structure that distinguishes trivial lifts from genuinely d-dimensional rules. A complementary criterion (no proper sub-rule reproduces dynamics on all admissibles) was tested empirically at $d = 5, 6$ and never failed when Definition F.2 was satisfied; we expect the two are equivalent for the rules in scope here, but have not proved this. A full $d = 7$ verification of the complementary criterion is pending.
2. *Why do* 6174 *and* 1746 *reappear as strict-d anchors at $d = 7$ but not at $d = 5, 6$?* The non-monotone pattern admits a candidate explanation in terms of the $d = 7$ rule space's allowance for three-position non-zero-sum tails, which the $d = 5, 6$ rule spaces do not accommodate. A precise statement and proof are open.
3. *Strict-anchor count at $d = 8, 9, \ldots$?* The $d = 7$ count is 11 — substantially larger than the $d = 6$ count of 4. A subsequent $d = 8$ scan in the canonical zero-padded multiset $\{7, 6, 4, 1, 0^4\}$ produced

89 strict anchors, and the symmetric pure-duplication multiset $\{7, 7, 6, 6, 4, 4, 1, 1\}$ produced 465. Whether the strict-anchor count continues to grow at $d \geq 9$, stabilizes, or oscillates is open.

4. *Parallel structure for the* $\{8, 5, 3, 2\}$*-thread.* Strict-d anchor counts at $d = 5, 6$ are $3, 3$ (versus $2, 4$ for $\{7, 6, 4, 1\}$). The asymmetric widths between threads, despite both threads sharing digit sum 18 at $d = 4$, is structural and not yet explained. The $d = 7$ analog for the $\{8, 5, 3, 2\}$-thread is pending.
5. *Are there other classical multiset cores beyond* $\{7, 6, 4, 1\}$ *and* $\{8, 5, 3, 2\}$*?* These are the only two universal multisets at $d = 4$. Whether the strict-anchor tower formalism extends to threads anchored at $d > 4$ (e.g., among the 33 universal fps at $d = 5$) is open.
6. *Relation to the main paper's Theorem 3.* The main paper proves 60714 universal at every $d \geq 5$. Under Definition F.2 of this appendix, 60714 is strict-universal at $d = 5, 6$ only; the lifts at $d \geq 7$ are non-strict. The main paper's theorem remains correct under its own (sv$=d$, F-or-0) framework. The reframing question — whether the central object of the theory is the (sv$=d$)-universal tower or the strict-d-universal tower — is not settled here.

14.7 Reproducibility note

The results of this appendix are reproducible from the supplementary scripts. The main scanner is `universality_scan_v3.py`; the independent verifier is `verifier_v2.py`. Both produce JSON outputs containing per-fp rule counts and the explicit list of strict-d rules for each anchor. Total wall-time on 2024-era commodity hardware: under three minutes for the complete $d \in \{5, 6, 7\}$ scan.

End of Appendix F.

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15 Appendix G. Classical-scope correction and confirmation of the cycle thread at $m = 4$

Status: Preliminary working notes added May 4–5, 2026. This appendix reports four findings developed after release of paper version 1.2.1: a methodological correction to the universality definition, re-verification of the paper's results under the corrected definition, empirical confirmation that the cycle thread continues to $m = 4$, and a refinement of the conjecture's lift mechanism.

None of the corrections invalidate prior results in the paper. They strengthen and extend them.

15.1 Purpose

The original verifier in this paper tested universality against an S-restricted input basin — that is, the set of non-repdigit d-digit multisets whose digits lie in the alphabet $S = \{0, 1, 4, 6, 7\}$. This restriction was inherited from the search infrastructure originally designed to *find* fixed points (whose digits naturally lie in S) but was inadvertently applied to *test* universality.

Classical Kaprekar universality, as the term is conventionally used in the literature on the digit-rearrangement function $K(n) = \text{sort\_desc}(n) - \text{sort\_asc}(n)$, requires the basin of K-iteration to include every non-repdigit d-digit input — that is, all multisets of digits drawn from $\{0, 1, \ldots, 9\}$ except repdigits.

This appendix (a) clarifies the distinction, (b) re-verifies the paper's results under the corrected definition, and (c) reports new empirical results that extend the cycle thread to $m = 4$.

15.2 Strict K-rules and classical universality

Throughout the paper, a K-rule at dimension d is a pair (π, σ) of permutations of $\{0, 1, \ldots, d-1\}$ giving the coefficient vector $c[i] = 10^{d-1-\pi^{-1}[i]} - 10^{d-1-\sigma^{-1}[i]}$ for $i = 0, \ldots, d-1$. The map $K_{\pi,\sigma}$ is then defined on a multiset $m_0 \geq m_1 \geq \cdots \geq m_{d-1}$ by $K(m) = |\sum_i c[i]\, m_i|$.

We restate the technical conditions used throughout the paper, with terminology pinned down as follows.

Definition G.1 *(strict K-rule).* A K-rule (π, σ) at dimension d is *strict* if $c[i] \neq 0$ for every $i \in \{0, \ldots, d-1\}$. Equivalently, $\pi^{-1}[i] \neq \sigma^{-1}[i]$ at every position. A non-strict rule is *deficient*: some position contributes nothing to $K(m)$.

Definition G.2 *(strict universal).* A fixed point F at dimension d is a *strict universal* if there exists a strict K-rule (π, σ) such that $K(F) = F$ and every S-admissible non-repdigit d-digit multiset converges to F under iteration of $K_{\pi,\sigma}$.

Definition G.3 *(classical universal).* A fixed point F at dimension d is a *classical universal* if there exists a strict K-rule under which every non-repdigit d-digit multiset (digits drawn from $\{0, 1, \ldots, 9\}$) converges to F.

Definition G.4 *(genuinely S-only).* A strict universal F is *genuinely S-only* if no strict K-rule for F is classical-universal. The terminology emphasizes that the property is intrinsic to F, not an artifact of which rule was sampled in the search.

Every classical universal is a strict universal. The converse may fail: a strict universal may be genuinely S-only. The two notions agree at $d = 4$ but diverge at higher d in a small minority of cases, as documented below.

15.3 Re-verification of v1.2.1 results under classical scope

A classical-scope verifier (described in §G.7 below) was applied to the strict universals catalogued in the paper. The results are summarized in Table G.1.

Table G.1. *Re-verification of the paper's published universal counts in classical scope.*

Dimension	Strict universals (paper v1.2.1)	Classical universals	Genuinely S-only
$d = 4$	2	2 (100%)	0
$d = 5$	33	33 (100%)	0
$d = 6$	506	confirmed; classical	0 known
$d = 8$, $m = 2$	481	**465** (96.7%)	**16** (3.3%)
$d = 12$, $m = 3$	55	**46** (83.6%)	**9** (16.4%)*

* *Upper bound: the $d = 12$ file recorded one K-rule per F. With multi-rule census (as performed at $d = 8$, where 64% of single-rule "S-only" universals admit a different rule that is classical), the genuine S-only count at $d = 12$ is expected to be lower.*

The 16 genuinely S-only universals at $d = 8$, $m = 2$ exhibit a striking pairing structure: they pair into multiset siblings, with each pair sharing the same digit multiset and differing by a digit transposition. For example: 16464717 and 16446717; 61771446 and 61477146; 66177144 and 66147714. The pattern is consistent with the genuinely S-only universals being structurally related rather than scattered. Their basin coverage in classical scope ranges from 99.1% to 99.9% — the missed inputs converge to alternative fixed points whose digit multisets lie outside S (typically containing digits 2, 3, 5, 8, or 9).

The principal result of the paper — that 60714 admits a coefficient- preserving lifting universal on $A_d \setminus E_d$ at every $d \geq 5$ — is unchanged under the classical-scope correction. All 60714 ladder universals are classical-universal at every tested $d \in \{5, 6, 7, 8, 9, 10, 11\}$.

15.4 Confirmation of the cycle thread at $m = 4$

A search at $d = 16$, $m = 4$ (multiset $\{7^4, 6^4, 4^4, 1^4\}$) was conducted using the candidates predicted by inserting a $\{7, 6, 4, 1\}$ block into known $d = 12$, $m = 3$ universals. The search produced 5 strict universals from 9,426 candidates passing distribution-alternation and density filters. Two of these are classical-universal:

Theorem G.1 *(empirical, $d = 16$, $m = 4$).* The fixed points

$$F_1 = 6{,}471{,}467{,}417{,}167{,}416, \qquad F_2 = 6{,}467{,}717{,}446{,}711{,}614$$

are each absorbed by every non-repdigit 16-digit multiset under their respective strict K-rules. Total basin: 2,042,965 multisets; iteration count for F_1: minimum 1, maximum 50, mean 25.9.

The remaining three strict universals at $d = 16$, $m = 4$ are genuinely S-only:

$$\{\, 6{,}474{,}646{,}711{,}711{,}476; \quad 6{,}467{,}714{,}467{,}117{,}416; \quad 6{,}467{,}716{,}714{,}471{,}146 \,\}$$

with classical basins of 2,042,708, 2,042,903, and 2,041,321 respectively. Each loses a small fraction of inputs (62 to 1,644 multisets out of 2,042,965) to alternative fixed points whose digit multisets lie completely outside S. The alternative fixed points include $F' = 6{,}576{,}637{,}809{,}610{,}377$ (multiset $\{0^3, 1, 3, 5, 6^3, 7^3, 8, 9\}$) and $F' = 6{,}468{,}613{,}468{,}216{,}515$ (multiset containing digits $\{1^3, 2, 3, 4^2, 5^2, 6^4, 8\}$).

Combined with the paper's existing results at $m = 1, 2, 3$, this confirms that the cycle thread $\{7^m, 6^m, 4^m, 1^m, 0^k\}$ at $d = 4m + k$ produces classical universals at every tested $m \in \{1, 2, 3, 4\}$.

15.5 Asymmetric lift behavior

Two cross-dimensional lift mechanisms were tested at the $d = 15 \to d = 16$ and $d = 12 \to d = 16$ transitions. The mechanisms behave differently:

Family B *(zero-insertion lift).* Predictions generated by inserting a single zero into the multiset of a known $d = 15$, $m = 3$, $k = 3$ universal, yielding $d = 16$, $m = 3$, $k = 4$ candidates. A Mac search of 731 candidates produced 0 **strict universals** and 0 algebraic K-fixed points.

Family D *(block-$\{7, 6, 4, 1\}$-insertion lift).* Predictions generated by inserting a length-4 block at various positions in $d = 12$, $m = 3$ universals, yielding $d = 16$, $m = 4$ candidates. A Mac search of 9,426 filter-passing candidates produced 5 **strict universals**, of which 2 are classical-universal (Theorem G.1 above).

The asymmetry refines the cycle conjecture's interpretation. The conjecture predicts that the multiset $\{7^m, 6^m, 4^m, 1^m, 0^k\}$ at $d = 4m + k$ contains classical universals; it does not predict that those universals are obtained by lifting universals from lower dimensions. None of the 5 universals at $d = 16$, $m = 4$ is the literal block-insertion of a known $d = 12$, $m = 3$ universal; rather, they are independently algebraic objects whose existence the block-lift heuristic correctly *predicts the existence of* (without identifying the specific arrangements).

The Family B negative result confirms that zero-insertion lifts do not produce universals as a generic mechanism. Apparent zero-insertion successes at lower dimensions (e.g., the 60714 ladder) succeed via coefficient-preserving algebra, not via simple structural insertion.

15.6 Distribution-alternation as a candidate filter

The distribution-alternation conjecture (paper §F.5) states that universal F values satisfy specific alternation patterns in their decimal digits. We tested this as a *predictive* filter at $d = 16$, $m = 4$.

Of the 9,426 candidates passing the distribution-alternation filter (density ≥ 0.75 and no Kaprekar-long-runs), 5 produced strict universals. Of the candidates *failing* the filter, none were tested (by design), so we cannot rule out additional universals among the 3,229 filtered-out candidates.

Among the 5 universals discovered:

- 1 is in the high-density (density = 1.0) bucket of 1,536 candidates: 0 of 1,536 universals — i.e., the high-confidence prediction returned 0 universals.
- 1 is in the medium-confidence bucket of 2,220 candidates: 1 of 2,220 = 0.045%.
- 4 are in the low-confidence bucket of 5,670 candidates: 4 of 5,670 = 0.071%.

This distribution is inverted from what the filter's confidence rankings predicted. The fine-grained predictor (density = 1.0, no Kaprekar-long-runs) did not identify a single universal at $d = 16$, $m = 4$. The four lower-density universals show F-arrangements with adjacent same-digit runs of length 2, contradicting the perfect-alternation hypothesis at this dimension.

We restate the predictor's empirical status:

- **Cell-level**: the cycle conjecture correctly identifies *which multisets* host universals at every m tested.
- **F-level within a cell**: the distribution-alternation filter is necessary (almost all universals satisfy it) but only $\sim 0.05\%$ specific (most filter-passing F values are not universal).
- **Fine-grained**: the conjunction "density = 1.0 + no Kaprekar-long-runs" is *not* a positive predictor at $d = 16$. The tightest signature predicted no universals; the actual universals appeared in less-restrictive buckets.

The conjecture's existence claim is supported. The conjecture's predictive content for individual F values requires further work.

15.7 Methods note: classical-scope verification

Three scripts were developed for the corrected verification:

1. `audit_full_basin.py` — given a single $(F, \pi^{-1}, \sigma^{-1})$, iterates K from every non-repdigit d-digit multiset and reports whether F is a classical universal. Caching makes this fast: at $d = 16$,

the full basin of 2,042,965 multisets is processed in approximately 13 seconds on a 2026-vintage laptop.
2. `reverify_classical_all_rules.py` — given a JSON of (F, rule) pairs, classifies each F as classical, genuinely S-only, or partial. For each F, all available strict rules are tested before declaring genuinely S-only.
3. `verify_d16_familyD_classical.py` — corrected production verifier for new searches, using full-basin scope rather than S-restricted scope.

The scripts are pure-stdlib Python, fully self-contained, and reproducible. They are deposited at the project's GitHub repository alongside the released paper.

15.8 Open questions raised by this appendix

1. **F-level prediction within a cell.** What property distinguishes the 5 strict universals from the other 9,421 distribution-alternation-passing candidates at $d = 16$, $m = 4$? Cycle structure does not separate them; density does not; Kaprekar-long-runs does not.
2. **Why the high-density predictor failed at $d = 16$.** The conjecture's tightest signature (density = 1.0, no Kaprekar-long-runs) returned zero universals at $d = 16$, $m = 4$, while less-restrictive buckets returned all five. What changes between $d = 12$ (where the high-density signature held) and $d = 16$ (where it does not)?
3. **Characterization of the 16 genuinely S-only universals at $d = 8$, $m = 2$.** They pair into multiset siblings. What algebraic property generates the pairing? Are they explained by a single mechanism distinct from the classical-universal majority?
4. **Whether the cycle thread is specific to the digit set $\{1, 4, 6, 7\}$.** Independent searches in other digit sets are in progress. Preliminary results (forthcoming) suggest that at least one other digit set produces classical universals at $m = 2$, with the $\{1, 4, 6, 7\}$ thread being notably denser at small m.
5. **Alternative attractors absorbing the missing inputs of genuinely S-only universals.** At $d = 16$, the 3 genuinely S-only universals lose inputs to fixed points like 6,576,637,809,610,377 whose digits include $\{0, 3, 5, 7, 8, 9\}$. Are these alternative fixed points themselves universals over their own multisets? This is testable.

15.9 Acknowledgments

The classical-scope correction was identified during ongoing work in 2026-05. The scope error and the corrected definitions are reported here for the first time.

The Mac runs reported in this appendix were performed on a 2026 Apple Silicon laptop using pure-Python implementations of K-iteration with caching, totaling approximately 9 hours of compute time (Family D verifier: 7.9 hours; reverification of $d = 8, 12, 16$ universals: 1.0 hour combined).

End of Appendix G.

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